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AN IMPLICIT RELATION FOR FOUR NON-SELF MAPPINGS IN PARTIAL METRIC SPACES

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ABSTRACT

Berinde and Vetro (2012) introduced fixed points theorems involving implicit relations for metric spaces and ordered metric spaces. These theorems were extended into partial metric spaces by other researchers. Vetro and Vetro studied coincidence point and common fixed point theorems for one pair of self-mappings satisfying conditions defined by implicit relations in the settings of partial metric space. We extend these results by Vetro and Vetro to two pairs of non-self mappings in complete partial metric spaces having a convex structure.

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1. Introduction and Preliminaries. Since the Banach contraction theorem described the conditions for a contractive mapping to have a fixed point in a complete metric spaces, researchers have proved several fixed point theorems of the form

$$d(Tx,Ty) \le F\big(d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(Tx,fy)\big), \qquad (1.1)$$

where f and T are self mappings in complete metric spaces. Theorems of this form abound also for non-self mappings in metric spaces and in generalizations of metric spaces such as the partial metric spaces. Examples of fixed point theorems of this form include Gajić and Rakoc'ević

[10] and Imdad and Kumar [12] for non-self mappings in complete metric spaces.

The expression (1.1) can be converted into the following form:
$F\left(d(Tx,Ty),d(fx,fy),d(fx,Tx),d(fy,Ty),d(fx,Ty),d(Tx,fy)\right)\leq 0.$

Here F is a function of form $F:\mathbb{R}_+^6 \to \mathbb{R}$, with specified properties that ensure the presence of a fixed point in a metric space (X,d) which contains the points x and y. When written in this way, the expression is called an implicit relation. Fixed point theorems involving implicit relations provide proofs for a large class of fixed point theorems written in the style of (1.1).

In this study, we will extend a fixed point theorem involving an implicit relation for partial metric spaces proved by Vetro and Vetro [19]. These researchers proved their theorem for a pair of self mappings. Here we generalize this theorem so that it applies for two pairs of non-self mappings in a metrically convex partial metric space.

We now introduce explanations which will be of use in this paper.

Definition 1.1 ([15]). *A partial metric on a non-empty set X is a mapping $p = X\times X \to [0,\infty)$, such that for all x, y, $z \in X$*

(P0): $0 \leq p(x,x) \leq p(x,y)$,

(P1): $x = y$ *if and only if* $p(x,x)= p(x,y) = p(y,y)$,

(P2): $p(x,y)=p(y,x)$ and

(P3): $p(x,y)\leq p(x,z)+p(z,y)-p(z,z)$.

The pair (X,p) is said to be a partial metric space.

From Definition 1.1, we deduce the following:

$$p(x, y) = 0 \text{ implies } x = y. \tag{1.2}$$

If $p(x,y)=0$, then $p(x,x)=0$ because $0 \leq p(x,x) \leq p(x,y)$ from (P0). Similarly, $p(x,y)=0$ implies $p(y,y)=0$ because $0 \leq p(y,y)\leq p(x,y)$. Hence, $p(x,y)=0$ implies $p(x,x)=p(x,y)=p(y,y)=0$. From (P1) this means that $x=y$.

From (P3), we infer that

$$p(x, y) \leq p(x, z) + p(z, y). \tag{1.3}$$

As an example, let $X=\mathbb{R}_+$ and let $p:\mathbb{R}_+^2 \to \mathbb{R}_+$, $p(x,y)=\max\{x,y\}$. Then (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon): x\in X, \varepsilon>0\}$ where

$B_p(x,\varepsilon)=\{y\in X: p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

Definition 1.2 [15] *Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ *converges to* $x\in X$ *if and only if* $p(x,x)=\lim_{n\to\infty} p(x,x_n)$.

(ii) $\{x_n\}$ *is called a Cauchy sequence if and only if there exists (and is finite)* $\lim_{n,m\to+\infty} p(x_n,x_m)$.

Definition 1.3. *A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$ such that*

$$p(x,x)=\lim_{n,m\to+\infty} p(x_n,x_m).$$

The definition of a 0-Cauchy sequence and a 0-complete partial metric is given here.

Definition 1.4 [15].

(i) *A sequence $\{x_n\}$ in a partial metric space (X,p) is said to be 0-Cauchy if* $\lim_{n,m\to\infty} p(x_n,x_m)=0$.

(ii) *A partial metric space (X,p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ converges in X with respect to τ_p, to a point $x\in X$ such that p(x,x)=0. In this case, p is a 0-complete partial metric on X.*

The following is the definition of a metric derived from a partial metric.

Lemma 1.5 [15] *If p is a partial metric on X, then the function $p^s: X\times X\to\mathbb{R}_+$ given by*

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y) \tag{1.4}$$

defines a metric on X.

A metrically convex metric space is defined as follows:

Definition 1.6 [2]. *A complete metric space (X,d) is said to be (metrically) convex if X has the property that for each $x,y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$, such that $d(x,z)+d(z,y)=d(x,y)$.*

If (X,d) is a metrically convex metric space, and $x,y \in X$, we term

$$seg[x,y] = \{z \in X : d(x,y) = d(x,z) + d(z,y)\}. \tag{1.5}$$

We get the following lemma from Assad and Kirk [2].

Lemma 1.7. *Let C be a closed subset of the complete and convex metric space X. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that*

$$d(x,z) + d(z,y) = d(x,y).$$

Using Lemma 1.5, we can rephrase Lemma 1.7 as follows

Lemma 1.8. *Let C be a closed subset of the complete and convex metric space X. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of C) such that $z \in seg[x,y]$.*

We introduce the metrically convex partial metric space.

Definition 1.9. *A partial metric space (X,p) is said to be metrically convex if the corresponding metric space (X,p^s) is metrically convex in the sense of* Lemma 1.5, where $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$.

As an example, the partial metric space $(\mathbb{R}_+, p)$ where $p(x,y)=\max\{x,y\}$ for all $x, y \in \mathbb{R}_+$ is metrically convex. This is because the metric space (X,p^s) is metrically convex, where $p^s(x,y)=|x-y|$, which is the metric derived from the partial metric p.

Lemma 1.10. *Let (X,p) be a metrically convex partial metric space. Let $x, y \in X$. If $z \in seg[x,y]$ then:*

(i) $p(x,y) - p(x,z) = p(z,y) - p(z,z)$,

(ii) $p(x,x) \geq p(x,z)$.

Proof. Applying (1.4) to Definition 1.6, if $z \in seg[x,y]$, then we have:

$$p^s(x,y) = p^s(x,z) + p^s(z,y)$$

$$\Rightarrow 2p(x,y)-p(x,x)-p(y,y)=2p(x,z)-p(x,x)-p(z,z)$$
$$+2p(z,y)-p(z,z)-p(y,y)$$
$$\Rightarrow p(x,y)=p(x,z)+p(z,y)-p(z,z)$$
$$\Rightarrow p(x,y)-p(x,z)=p(z,y)-p(z,z).$$

As $p(x,y)-p(z,z)\geq 0$, from (P0) of Definition 1.1 we have

$$p(x,y)\geq p(x,z).$$

Lemma 1.11. *Let C be a non-empty subset of a metrically convex partial metric space (X,p) which is closed in (X,p^s). If $x\in C$ and $y\in X\setminus C$, then there exists a point $z\in\partial C$ (the boundary of C with respect to (X,p^s)) such that*

$$p(x,y)=p(x,z)+p(z,y)-p(z,z).$$

Proof. From Definition 1.9, if the partial metric space (X,p) is metrically convex, then (X,p^s) is metrically convex. From Lemma 1.7, this means that if $x\in C$ and $y\in X\setminus C$, then there exists z in ∂C, (the boundary of C), such that $p^s(x,y)=p^s(x,z)+p^s(z,y)$. Using (1.4), this means

$$p^s(x,y)=p^s(x,z)+p^s(z,y)$$
$$\Rightarrow 2p(x,y)-p(x,x)-p(y,y)=2p(x,z)-p(x,x)-p(z,z)+2p(z,y)-p(z,z)-p(y,y)$$
$$\Rightarrow 2p(x,y)=2p(x,z)+2p(z,y)-2p(z,z)$$
$$\Rightarrow p(x,y)+p(z,z)=p(x,z)+p(z,y)$$
$$\Rightarrow p(x,z)+p(z,y)=p(x,y)+p(z,z).$$

We prove the following lemma, which is modified from Theorem 1 of Assad and Kirk [2], and is necessary for our work.

Lemma 1.12. *Consider a sequence $\{w_n\}_{n\in N}\in\mathbb{R}_+$ such that, for all $n\geq 2$ we have*

$$w_n\leq k\max\{w_{n-2},w_{n-1}\},k\in(0,1), \tag{1.6}$$

then

$$w_n\leq k^{n/2}k^{-1/2}\max\{w_0,w_1\}. \tag{1.7}$$

Proof. We prove the lemma by induction. First we show that it holds for $n=2$.

We note that $k \in (0,1)$ implies $k \leq k^{1/2}$. Hence, if $n=2$, then (1.6) leads to

$$w_2 \leq k \max\{w_0, w_1\} \leq k^{1/2} \max\{w_0, w_1\} = k^{2/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.8}$$

Now we show that the lemma holds for $n=3$. If $n=3$, then (1.6) leads to $w_3 \leq k \max\{w_1, w_2\}$. If $w_1 \geq w_2$, then we get

$$w_3 \leq k \max\{w_1, w_2\}$$

$$\Rightarrow w_3 \leq kw_1 \leq k \max\{w_0, w_1\} = k^{3/2}.k^{-1/2} \max\{w_0, w_1\}.$$

If however $w_1 < w_2$, we get

$$w_3 \leq k \max\{w_1, w_2\}$$

$$\Rightarrow w_3 \leq kw_2$$

$$\Rightarrow w_3 \leq k \times k^{2/2} k^{-1/2} \max\{w_0, w_1\}. \text{ from (1.8)}$$

$$\leq k^{3/2} \max\{w_0, w_1\}$$

$$\leq k^{3/2} k^{-1/2} \max\{w_0, w_1\}, \text{ because } k^{-1/2} \geq 1.$$

We now show that, if Lemma 1.12 holds for $1 \leq n \leq j$ where $j \geq 3$, then it must be hold for $j+1$. Hence we have from (1.6)

$$w_{j+1} \leq k \max\{w_{j-1}, w_j\}. \tag{1.9}$$

We consider two cases.

Case (i): Suppose $w_{j-1} \leq w_j$. Then (1.9) becomes

$$w_{j+1} \leq kw_j$$

$$\leq k.k^{j/2} k^{-1/2} \max\{w_0, w_1\} \text{ from (1.7)}$$

$$= k^{(j+2)/2} k^{-1/2} \max(w_0, w_1). \tag{1.10}$$

Case (ii): Suppose $w_{j-1} > w_j$. Then (1.9) becomes

$$w_{j+2} \leq kw_{j-1}$$

$$\leq k.k^{(j-1)/2} k^{-1/2} \max\{w_0, w_1\} \text{ from (1.7)}$$

$$= k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.11}$$

We note that for $j \geq 3$ and $k \in (0,1]$ we have $k^{(j+1)/2} \geq k^{(j+2)/2}$. Hence, (1.10) and (1.11) imply that

$$w_{j+1} \le k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}.$$

We make modifications to the description of the set of implicit relations as defined by Berinde [8]. Let $\mathcal{F}$ be a family of all continuous real functions $F:\mathbb{R}_+^6 \to \mathbb{R}$ and the following conditions:

(F_{1A}): F is non-increasing in the fifth variable and $F(u_1, v_1, v_2, u_2, u_1 + v_2, v_2 - v_1) \le 0$ for $u_1, u_2 \ge 0, u_2 \le v_2 + u_1$ and $0 \le v_1 \le v_2$ implies that there exists $h \in [0,1/2)$ such that $u_1 \le hv_2$.

(F2): $F(u,u,0,0,u,u) > 0$, for all $u>0$.

We define the property (F_{1a}) as a special case of property (F_{1A}), with $u=u_1=u_2$, $v=v_1=v_2$. The condition F_{1a} hence reads

(F_{1a}): F is non-increasing in the fifth variable and $F(u,v,v,u,u+v, 0) \le 0$ for $u \ge 0$ implies that there exists $h \in [0,1/2)$ such that $u_1 \le hv_2$.

Example 1.13. The following functions $F \in \mathcal{F}$ satisfy the properties (F_{1A}) and (F_2).

(i) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2$, where $a \in [0,1/2)$;

(ii) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - b(t_3 + t_4)$, where $t_1 \ge t_4$ and $b \in [0,1/3)$;

(iii) $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - c(t_5 + t_6)$, where $c \in [0,1/5)$.

Jungck [14] provided the following definitions:

Definition 1.14 [14] Let C be a non-empty subset of X and $f,T:C\to X$.
A point $x \in C$ is called a coincidence point of f and X if $Tx= fx$.

Definition 1.15 [14]The mappings T and f are said to be coincidentally commuting if they commute at their coincidence point, that is, $Tfx=fTx$ whenever $Tx=fx$.

The aim of this research is to extend the following theorem by Vetro and Vetro into non-self mappings. Note that in their theorem F_{1a} is defined with $h \in [0, 1)$.

Theorem 1.16 ([19]) Let (X, p) be a partial metric space and $T, f: X \to X$ be self-mappings such that $TX \subseteq fX$. Assume that there exists $F \in \mathcal{F}$ satisfying (F_{1a}) such that, for all $x,y \in X$, the following condition holds:

$$F\big(p(Tx,Ty),p(fx,fy),p(fx,Tx),p(fy,Ty),p(fx,Ty),p(fy,Tx)-p(fy,fy)\big)\leq 0. \tag{1.12}$$

If fX is a 0-complete subspace of X, then T and f have a coincidence point. Moreover, if T and f are weakly compatible and F satisfies also (F_2), then T and f have a unique common fixed point. Further, for any $x_0 \in X$, the T–f–sequence $\{Tx_n\}$ with initial point x_0 converges to the common fixed point.

2 Main Result. We start by making some modification to Lemma 2.1 by Vetro and Vetro [19] which will then be used to develop Theorem 2.2.

Lemma 2.1. Let C be a non-empty subset of a partial metric space (X,p) and $f,g,S,T:C\to X$ be non-self mappings. Assume that there exists $F\in\mathcal{F}$ satisfying (F_{1A}) such that, for all $x, y\in C$, we have

$$F\big(p(Sx,Ty),p(fx,gy),p(fx,Sx),p(gy,Ty),p(fx,Ty),p(gy,Sx)-p(gy,gy)\big)\leq 0. \tag{2.1}$$

Then, for all $z\in X$ such that $fz= gz = Tz$, we have

$p(fz,fz)=p(gz,gz)=p(Sz,Sz)=p(Tz,Tz)=0.$

Proof. Let $x = y = z$ in equation (2.1). We get

$$F\big(p(Sz,Tz),p(fz,gz),p(fz,Sz),p(gz,Tz),p(fz,Tz),p(gz,Sz)-p(gz,gz)\big)\leq 0$$
$$\Rightarrow F\big(p(Sz,gz),p(gz,gz),p(gz,Sz),p(gz,gz),p(gz,gz),p(gz,Sz)-p(gz,gz)\big)\leq 0. \tag{2.2}$$

Because of F_{1A}, and setting $u_1=p(Sz,gz)$, $u_2=p(gz,gz)$ and $v_1=p(gz,gz)$, $v_2=p(gz,Sz)$, (2.2) implies

$F(u_1,v_1,v_2,u_2,u_1+v_2,v_2-v_1)\leq 0,$

as F is non-increasing in the fifth variable.

$\Rightarrow p(Sz,gz)\leq hp(gz,Sz)=hp(Sz,gz)$, for $h\in[0,1/2)$

$\Rightarrow p(Sz,gz)=0$

$\Rightarrow Sz=gz$, by (1.2)

$\Rightarrow p(fz,fz)=p(gz,gz)=p(Sz,Sz)=p(Tz,Tz)=0.$

Now we deal with our main theorem.

Theorem 2.2. Let (X,p) be a complete metrically convex partial metric space and C a non empty closed subset of X, the closure being with respect to (X,p^s). Let ∂C, the boundary of C with respect to (X,p^s), be non empty. Let $f,g,S,T:C\to X$ satisfy the following conditions:

(i) $fx\in \partial C$ implies $Sx\in C$, $gx\in \partial C$ implies $Tx\in C$,

(ii) $\partial C\subset fC\cap gC$,

(iii) $SC\cap C\subset gC, TC\cap C\subset fC$,

(iv) fC or gC is a 0-complete subspace of (X,p) and

(v) f and g are continuous mappings.

Assume that there exists $F\in \mathcal{F}$ satisfying (F_{1A}) such that, for all $x,y\in X$, condition (2.1) holds. Then f,g,S and T have a coincidence point. Moreover, if the pairs (f,S) and (g,T) are coincidentally commuting and F satisfies also (F_2), then f,g,S and T have a unique common fixed point.

Proof. Commencing with an arbitrary point $w\in \partial C$. From (ii), we can find $x_0\in C$ such that $fx_0=w$. From (i), we have $Sx_0\in C$. According to (iii), we can choose $x_1\in C$ such that $gx_1=Sx_0$.

Then, we find Tx_1.

We proceed to generate the sequences $\{x_n\},\{fx_{2n}\},\{gx_{2n+1}\},\{Sx_{2n}\}$ and $\{T\,x_{2n+1}\}$ inductively as follows:

If $Sx_{2n}\in C$, then by (iii) we can choose $x_{n+1}\in C$ such $gx_{2n+1}=Sx_{2n}$. If however $Sx_{2n}\notin C$ and $n\geq 1$, then by Lemma 1.8, there exists a $w_{2n+1}\in \partial C$ such that $w_{2n+1}\in seg[Tx_{2n-1}.Sx_{2n}]$. Hence, by (ii), we can choose $x_{n+1}\in C$ such that $fx_{2n+1}=w_{2n+1}$.

Then, we find Tx_{2n+1}

Similarly, if $Tx_{2n+1}\in C$, then by (iii) we can choose $x_{2n+2}\in C$ such that $fx_{2n+2}=Tx_{2n+1}$.

However, if $Tx_{2n+1}\notin C$, then by Lemma 1.8, there exists a $w_{2n+2}=\in \partial C$ such that $w_{2n+2}\in seg[Sx_{2n},Tx_{2n+1}]$. Hence, by (ii), we can choose $x_{2n+2}\in C$ such that $fx_{2n+2}=w_{2n+2}$. Then, we find Sx_{n+2}.

We claim that

$$fx_{2n+2} \neq Tx_{2n+1} \Rightarrow gx_{2n+1} = Sx_{2n} \tag{2.3}$$

Suppose we have $gx_{2n+1} \neq Sx_{2n}$. Then we have $gx_{2n+1} \in \partial C$, which by (i) means $Tx_{2n+1} \in C$. This implies, by the construction of the sequence, that $fx_{2n+2} = Tx_{2n+1}$, which is a contradiction. Thus we have proved (2.3).

In a similar manner, we can show that, for $n \geq 1$

$$gx_{2n+1} \neq Sx_{2n} \Rightarrow fx_{2n} = Tx_{2n-1}. \tag{2.4}$$

We now partition the sequence $\{x_n\}$ into subsequences P and Q with

$$P = \{x_{2n} : fx_{2n} = Tx_{2n-1}, n \geq 1\} \cup \{x_{2n+1} : gx_{2n+1} = Sx_{2n}\}$$

and

$$Q = \{x_{2n} : fx_{2n} \neq Tx_{2n-1}, n \geq 1\} \cup \{x_{2n+1} : gx_{2n+1} \neq Sx_{2n}\}.$$

In (2.1), let us set $x = x_{2n}, y = x_{2n+1}$. We get

$$F\big(p(Sx_{2n}, Tx_{2n+1}), p(fx_{2n}, gx_{2n+1}), p(fx_{2n}, Sx_{2n}), p(gx_{2n+1}, Tx_{2n+1}),$$
$$p(fx_{2n}, Tx_{2n+1}), p(gx_{2n+1}, Sx_{2n}) - p(gx_{2n+1}, gx_{2n+1})\big) \leq 0. \tag{2.5}$$

We consider the following cases.

Case 1: Let $(x_{2n+1}, x_{2n+2}) \in P \times P$. In this case, we $(gx_{2n+1}, fx_{2n+2}) = (Sx_{2n}, Tx_{2n+1})$. Applying (2.5), we get

$$F\big(p(gx_{2n+1}, fx_{2n+2}), p(fx_{2n}, gx_{2n+1}), p(fx_{2n}, gx_{2n+1}), p(gx_{2n+1}, fx_{2n+2}),$$
$$p(fx_{2n}, fx_{2n+2}), p(gx_{2n+1}, gx_{2n+1}) - p(gx_{2n+1}, gx_{2n+1})\big) \leq 0.$$

If we set $u = p(gx_{2n+1}, fx_{2n+2}), v = p(fx_{2n}, gx_{2n+1})$, we get

$$F\big(u, v, v, u, p(fx_{2n}, fx_{2n+2}), 0\big) \leq 0. \tag{2.6}$$

Using (1.2), we have

$$p(fx_{2n}, fx_{2n+2}) \leq p(fx_{2n}, gx_{2n+1}) + p(gx_{2n+1}, fx_{2n+2}) = v + u.$$

As F has the F_{1a} property which states that F is non-increasing in the fifth variable, (2.6) implies

$$F(u, v, v, v+u, 0) \leq 0,$$

which implies implies there is $h \in [0, 1)$ such that $u \leq hv$. This means for $(x_{2n+1}, x_{2n+2}) \in P \times P$, we have

$$p(gx_{2n+1}, fx_{2n+2}) \leq hp(fx_{2n}, gx_{2n+1}). \tag{2.7}$$

Using a similar argument with $(x_{2n}, x_{2n+1}) \in P \times P, n \geq 1$, we deduce that there is $h \in [0,1)$ such that

$$p(fx_{2n}, gx_{2n+1}) \leq hp(gx_{2n-1}, fx_{2n}). \tag{2.8}$$

Case 2. We now consider the case where $(x_{2n+1}, x_{2n+2}) \in P \times Q$. From the construction of proof, this means $gx_{2n+1} = Sx_{2n}$. We apply (2.5) and obtain

$$F\big(p(gx_{2n+1}, Tx_{2n+1}), p(fx_{2n}, gx_{2n+1}), p(fx_{2n}, gx_{2n+1}), p(gx_{2n+1}, Tx_{2n+1}),$$
$$p(fx_{2n}, Tx_{2n+1}), p(gx_{2n+1}, gx_{2n+1}) - p(gx_{2n+1}, gx_{2n+1})\big) \leq 0$$

Let $u = p(gx_{2n+1}, Tx_{2n+1}), v = p(fx_{2n}, gx_{2n+1})$. Note that from (1.2), we have

$$p(fx_{2n}, Tx_{2n+1}) \leq p(fx_{2n}, gx_{2n+1}) + p(gx_{2n+1}, Tx_{2n+1}) = v + u.$$

We also use the F_{1a} property that states that F is non-increasing in the fifth variable. We get

$$F(u, v, v, u, v+u, 0),$$

which implies for some $h \in [0, 1)$, we have

$$u \leq hv$$
$$\Rightarrow p(gx_{2n+1}, Tx_{2n+1}) \leq hp(fx_{2n}, gx_{2n+1}).$$

Because $fx_{2n+2} \in seg[Sx_2, Tx_{2n+1}] = seg[gx_{2n+1}, Tx_{2n+1}]$, we deduce from Lemma 1.10 (ii), that

$$p(gx_{2n+1}, fx_{2n+2}) \leq p(gx_{2n+1}, Tx_{2n+1}) \leq hp(fx_{2n}, gx_{2n+1}) . \tag{2.9}$$

Using a similar argument with $(x_{2n}, x_{2n+1}) \in P \times Q, n \geq 1$, we deduce that there is $h \in [0, 1)$ such that

$$p(fx_{2n}, gx_{2n+1}) \leq p(fx_{2n}, Sx_{2n}) \leq hp(gx_{2n-1}, fx_{2n}). \tag{2.10}$$

Case 3. We consider when $(x_{2n+1}, x_{2n+2}) \in Q \times P$. In this case, we have $gx_{2n+1} \in seg[Tx_{2n-1}, Sx_{2n}], fx_{2n+2} = Tx_{2n+1}$ and $fx_{2n} = Tx_{2n-1}$ (see (2.4)).

Applying (1.3), we have

$$p(gx_{2n+1},fx_{2n+2}) \le p(gx_{2n+1},Sx_{2n}) + p(Sx_{2n},fx_{2n+2})$$
$$\le 2\max\{p(gx_{2n+1},Sx_{2n}),p(Sx_{2n},fx_{2n+2})\} \tag{2.11}$$

For $p(gx_{2n+1},Sx_{2n}) \ge p(Sx_{2n},fx_{2n+2})$, (2.11) becomes

$$\begin{aligned} p(gx_{2n+1},fx_{2n+2}) &\le 2p(gx_{2n+1},Sx_{2n}) \\ &\le 2p(Tx_{2n-1},Sx_{2n}), \text{ using Lemma 1.10 (ii)} \\ &= 2p(fx_{2n},Sx_{2n}) \\ &\le 2hp(gx_{2n-1},fx_{2n}), \end{aligned} \tag{2.12}$$

using the argument of Case 2, (see(2.10)).

Suppose $p(gx_{2n+1},Sx_{2n}) < p(Sx_{2n},fx_{2n+2})$. Then (2.11) becomes

$$p(gx_{2n+1},fx_{2n+2}) \le 2p(Sx_{2n},fx_{2n+2}). \tag{2.13}$$

Let us deal with $p(Sx_{2n},fx_{2n+2})$. Consider (2.1) with $x = x_{2n}, y = x_{2n+1}$. Then, we have

$$F\big(p(Sx_{2n},Tx_{2n+1}),p(fx_{2n},gx_{2n+1}),p(fx_{2n},Sx_{2n}),p(gx_{2n+1},Tx_{2n+1}),$$
$$p(fx_{2n},Tx_{2n+1}),p(gx_{2n+1},Sx_{2n})-p(gx_{2n+1},gx_{2n+1})\big) \le 0$$
$$\Rightarrow F\big(p(Sx_{2n},fx_{2n+2}),p(fx_{2n},gx_{2n+1}),p(fx_{2n},Sx_{2n}),p(gx_{2n+1},fx_{2n+2}),$$
$$p(fx_{2n},fx_{2n+2}),p(gx_{2n+1},Sx_{2n})-p(gx_{2n+1},gx_{2n+1})\big) \le 0. \tag{2.14}$$

As $gx_{2n+1} \in seg[Tx_{2n-1},Sx_{2n}] = seg[fx_2,Sx_{2n}]$, we have from Lemma 1.10

(i) $p(fx_{2n},Sx_{2n}) - p(fx_{2n},gx_{2n+1}) = p(gx_{2n+1},Sx_{2n}) - p(gx_{2n+1},gx_{2n+1})$.

Hence, (2.14) becomes

$$F\big(p(Sx_{2n},fx_{2n+2}),p(fx_2,gx_{2n+1}),p(fx_{2n},Sx_{2n}),p(gx_{2n+1},fx_{2n+2}),$$
$$p(fx_{2n},fx_{2n+2}),p(fx_{2n},Sx_{2n})-p(fx_{2n},gx_{2n+1})\big) \le 0. \tag{2.15}$$

Let $u_1 = p(Sx_{2n},fx_{2n+2}), u_2 = p(gx_{2n+1},fx_{2n+2}), v_1 = p(fx_{2n},gx_{2n+1})$ and $v_2 = p(fx_{2n},Sx_{2n})$. We note that by (1.3), we have

$$p(fx_{2n},fx_{2n+2}) \le p(fx_{2n},Sx_{2n}) + p(Sx_{2n},fx_{2n+2}) = v_2 + u_1.$$

Using Lemma 1.10 (ii), because $gx_{2n+1} \in seg[fx_{2n},Sx_{2n}]$, we have $p(fx_{2n}, gx_{2n+1}) \leq p(fx_{2n}, Sx_{2n})$ implying $v_1 \leq v_2$.

As required by the assumption, we also have $u_2 \leq v_2 + u_1$. This is because

$$\begin{aligned} u_2 &= p(gx_{2n+1}, fx_{2n+2}) \\ &\leq p(gx_{2n+1}, Sx_{2n}) + p(Sx_{2n}, fx_{2n+2}), \text{ by (1.3)} \\ &\leq p(fx_{2n}, Sx_{2n}) + p(Sx_{2n}, fx_{2n+2}), \text{ by Lemma 1.10 (ii)} \\ &= v_2 + u_1. \end{aligned}$$

Using the property (F_{1A}) that F is non-increasing in the fifth variable, (2.15) implies

$$F(u_1, v_1, v_2, u_2, v_2 + u_1, v_2 - v_1) \leq 0$$

$$\Rightarrow u_1 \leq hv_2$$

$$\Rightarrow p(Sx_{2n}, fx_{2n+2}) \leq hp(fx_{2n}, Sx_{2n}). \tag{2.16}$$

We continue with (2.13).

$$\begin{aligned} p(gx_{2n+1}, fx_{2n+2}) &\leq 2p(Sx_{2n}, fx_{2n+2}) \\ &\leq 2hp(fx_{2n}, Sx_{2n}), \text{ from (2.16)} \\ &\leq 2h \times hp(gx_{2n-1}, fx_{2n}) \text{ using the argument of Case 2, (see(2.10))} \\ &\leq hp(gx_{2n-1}, fx_{2n}), \text{because } 2h < 1. \qquad (2.17) \end{aligned}$$

Hence, from (2.12) and (2.17), we conclude that for $(x_{2n+1}, x_{2n+2}) \in Q \times P$, we have

$$p(gx_{2n+1}, fx_{2n+2}) \leq 2hp(gx_{2n-1}, fx_{2n}). \tag{2.18}$$

We get the following similar result if we argue with $(x_{2n}, x_{2n+1}) \in Q \times P$:

$$p(fx_{2n}, gx_{2n+1}) \leq 2hp(fx_{2n-2}, gx_{2n-1}). \tag{2.19}$$

We note that $x_{2n+1}, x_{2n+2} \in Q \times Q$ is not possible.

Let us define the sequence $\{w_n\}$ as $w_{2n} = fx_{2n}, w_{2n+1} = gx_{2n+1}$ for all $n \in$.
Thus from (2.7), (2.8), (2.9), (2.10), (2.18) and (2.19), we get for all $n \geq 1$

$$p(w_n, w_{n+1}) \leq 2h \max\{p(w_{n-2}, w_{n-1}), p(w_{n-1}, w_n)\} . \tag{2.20}$$

As $2h < 1$, according to Lemma 1.12, (2.20) implies

$$p(w_n, w_{n+1}) \leq (2h)^{n/2} \delta, \tag{2.21}$$

where $\delta = (2h)^{-1/2} \max\{p(w_0,w_1), p(w_1,w_2)\}$.

Consider $n,m \in$ with $n>m$. Then, from (1.3) by induction, we have

$$\begin{aligned} p(w_m,w_n) &\le \sum_{i=m}^{n-1} p(w_i,w_{i+1}) \\ &\le \sum_{i=m}^{n-1} (2h)^{i/2}\delta \\ &\le \delta\sum_{i=m}^{\infty}(2h)^{i/2} \\ &= \delta\frac{(2h)^{m/2}}{1-(2h)^{1/2}}, \text{ (sum of a Geometric Progression.)} \end{aligned}$$

As $m,n \to \infty$ we get

$$\lim_{m,n\to\infty} p(w_m,w_n) = 0 < +\infty \tag{2.22}$$

This makes $\{w_n\}$ a 0-Cauchy sequence.

Consider the subsequence $\{w_{2n+1}\}$ of $\{w_n\}$. As $\{w_n\}$ a 0-Cauchy sequence, we have also $\{w_{2n+1}\} = \{gx_{2n+1}\}$ being a 0-Cauchy sequence.

Suppose gC is 0-complete. This means that there is $c \in gC$ such that

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} w_{2n+1} = \lim_{n\to\infty} gx_{2nK+1} = c \text{ and } p(c,c) = 0 \tag{2.23}$$

Consider the subsequence $\{gx_{2n_k+1}\}$ of $\{gx_{2n+1}\}$ such that $gx_{2n_k+1} = Sx_{2n_k}$. If we take $k \to +\infty$, we get $\lim_{n\to+\infty} gx_{2n_k+1} = \lim_{n\to+\infty} Sx_{2n} = c$, which implies

$$\lim_{n\to+\infty} gx_{2n+1} = \lim_{n\to+\infty} Sx_{2n} = c. \tag{2.24}$$

Because $w_n \to c$, this means that the subsequence $w_{2n} = fx_{2n} \to c$ also.

Now let us consider the subsequence $\{fx_{2n_k}\}$ of $\{fx_{2n}\}$ such that

$fx_{2n_k} = Tx_{2n_k-1}, n_k \ge 1$. Taking $k \to +\infty$, we get $\lim_{n\to+\infty} fx_{2n_k} = \lim_{n\to+\infty} Tx_{2n_k-1} = c$.

This implies

$$\lim_{n\to+\infty} fx_{2n} = \lim_{n\to+\infty} Tx_{2n-1} = c. \tag{2.25}$$

From (2.24) and (2.25), we get

$$\lim_{n\to+\infty} fx_{2n} = \lim_{n\to+\infty} Tx_{2n-1} = \lim_{n\to+\infty} gx_{2n+1} = \lim_{n\to+\infty} Sx_{2n} = c. \tag{2.26}$$

We get the same result (2.26) if we assume fC is 0-complete.

From (v) in the assumption, f and g are continuous mappings. Hence, (2.26) leads to

$$fz = gz = c, \text{ where } z = \lim_{n\to+\infty} x_n \tag{2.27}$$

We claim that z is a coincidence point of f and T.

Let $x = x_{2n}, y = z$ in (2.1). Using (2.27), we get

$$F\left(p(Sx_{2n},Tz), p(fx_{2n},gz), p(fx_{2n},Sx_{2n}), p(gz,Tz),\right.$$

$$\left.(fx_{2n},Tz), p(gz,Sx_{2n}) - p(gz,gz)\right) \le 0.$$

$$\Rightarrow F\left(p(Sx_{2n},Tz), p(fx_{2n},c), p(fx_{2n},Sx_{2n}), p(c,Tz),\right.$$

$$\left.(fx_{2n},Tz), p(c,Sx_{2n}) - p(c,c)\right) \le 0.$$

Because F is continuous, if we take $n \to +\infty$ and apply (2.26), we get

$$F\left(p(c,Tz), p(c,c), p(c,c), p(c,Tz), (c,Tz), p(c,c) - p(c,c)\right) \le 0$$

$$\Rightarrow F\left(p(c,Tz), 0, 0, p(c,Tz), 0\right) \le 0. \tag{2.28}$$

We use the property (F_{1a}) with $u = p(c,Tz)$, $v = 0$ and deduce that for some $h \in [0,1/2)$, we have

$u \le hv$

$\Rightarrow p(c,Tz) \le 0$

$\Rightarrow p(c,Tz) = 0$, by (P0) of Definition 1.1

$\Rightarrow Tz = c$, by (1.2).

Taking note of (2.27) we get

$$Tz = gz = fz = c. \tag{2.29}$$

Because the pair (f, T) is coincidentally commuting at z, we have

$$Tfz = fTz \Rightarrow Tc = fc. \tag{2.30}$$

We now claim that z is a coincidence point of g and S.

Let $x = z$, $y = x_{2n}$ in (2.1). Then, we have

$$F\left(p(Sz,Tx_{2n}), p(fz,gx_{2n}), p(fz,Sz), p(gx_{2n},Tx_{2n}),\right.$$

$$\left.p(fz,Tx_{2n}), p(gx_{2n},Sz) - p(gx_{2n},gx_{2n})\right) \le 0.$$

We take $n \to +\infty$. We use (2.26) and note by (2.29) that $fz = c$, and get

$$F\left(p(Sz,c), p(c,c), p(c,Sz), p(c,c), p(c,c), p(c,Sz) - p(c,c)\right) \le 0$$

$$\Rightarrow F\left(p(Sz,c), 0, p(c,Sz), 0, 0, p(c,Sz)\right) \le 0, \text{ because } p(c,c) = 0$$

$$\Rightarrow F\left(p(c,Sz),0,p(c,Sz),0,0,p(c,Sz)\right) \le 0. \tag{2.31}$$

If we denote $u_1=v_2=p(c,Sz)$ and $u_2=v_1=0$, and consider that F is non-increasing in the fifth variable, (2.31) becomes

$F(u_1,v_1,v_2,u_2,u_1+v_2,v_2-v_1) \le 0$

$\Rightarrow u_1 \le hv_2$, where $h \in [0,1/2)$

$\Rightarrow p(c,Sz) \le hp(c,Sz)$

$\Rightarrow p(c,Sz)=0,$

$\Rightarrow c=Sz$, by (1.2). $\quad$ (2.32)

Applying (2.29) to the above result we get $Sz=gz=c$, which shows that z is a coincidence point of g and S. Using the coincidental commuting property of g and S, we get

$$Sz=gz \Rightarrow gSz = Sgz \Rightarrow gc = Sc. \tag{2.33}$$

We claim that c is a common fixed point of f,g,S and T.

Let $x=c$ and $y=c$ in (2.1). From (2.30) and (2.33) we note that $fc=Tc$ and $gc=Sc$. Then, we have

$$F\left(p(Sc,Tc),p(fc,gc),p(fc,Sc),p(gc,Tc),p(fc,Tc),p(gc,Sc)-p(gc,gc)\right) < 0.$$

$$\Rightarrow F\left(p(gc,fc),p(fc,gc),p(fc,gc),p(gc,fc),p(fc,fc),p(gc,gc)-p(gc,gc)\right) \le 0.$$

$$\Rightarrow F\left(p(gc,fc),p(gc,fc),p(gc,fc),p(gc,fc),p(fc,fc),0\right) \le 0.$$

We set $u=v=p(gc,fc)$. We also note that, by P0 of Definition 1.1, $p(fc,fc) \le 2p(gc,fc)$. We consider the fact that F is non-increasing in the fifth variable and get

$F(u,v,v,u,u+v,0) \le 0$

$\Rightarrow u \le hv, h \in [0,1/2)$

$\Rightarrow p(gc,fc) \le hp(gc,fc)$

$\Rightarrow p(gc,fc) = 0$

$\Rightarrow gc = fc,$ by(1.2)

$\Rightarrow gc = gc = Tc,$ (2.30). $\quad$ (2.34)

By Lemma 2.1, as $fc=gc=Tc$, we have

$$p(fc,fc) = p(gc,gc) = p(Sc,Sc) = p(Tc,Tc) = 0. \tag{2.35}$$

Now consider (2.1) with $x=z$, $y=c$. Then, we have

$$F\left(p(Sz,Tc),p(fz,gc),p(fz,Sz),p(gc,Tc),p(fz,Tc),p(gc,Sz)-p(gc,gc)\right)\leq 0. \quad (2.36)$$

We note that $Sc=gc$ (by (2.33)) implies that $fc=gc=Sc=Tc$ by (2.30).

We recall that $fz=Sz=c$ from (2.29) and (2.32).

Thus (2.36) becomes

$$F\left(p(c,gc),p(c,gc),p(c,c),p(gc,gc),p(c,gc),p(gc,c)-p(gc,gc)\right)\leq 0. \quad (2.37)$$

We also note that $p(c,c)=0$ and $p(gc, gc)=0$ from (2.35). Hence, (2.37) becomes

$$F\left(p(c,gc),p(c,gc),0,0,p(c,gc),p(c,gc)\right)\leq 0. \quad (2.38)$$

Suppose $p(c,gc)>0$. Because F has the (F_2) property, this means

$$F\left(p(c,gc),p(c,gc),0,0,p(c,gc),p(c,gc)\right)>0.$$

This is a contradiction with (2.38). Hence,

$$p(c,gc) = 0 \Rightarrow gc=c, \text{ by } (1.2).$$

By (2.30) and (2.33), this implies

$$fc=gc=Sc=Tc=c,$$

which shows that c is a common fixed point of f, g, S and T, with $p(c,c)=0$.

Now, we will show that the common fixed point c is unique.

Suppose c' is also a common fixed point of f, g, S and T, and $p(c',c')=0$. We set $x=c, y=c'$ in (2.1). We get

$$F\left(p(Sc,Tc'),p(fc,gc'),p(fc,Sc),p(gc',Tc'),\right.$$

$$\left.p(fc,Tc'),p(gc',Sc)-p(gc',gc')\right)\leq 0$$

$$\Rightarrow F\left(p(c,c'),p(c,c'),p(c,c),p(c',c'),p(c,c'),p(c',c)-p(c',c')\right)\leq 0$$

$$\Rightarrow F\left(p(c,c'),p(c,c'),0,0,p(c,c'),p(c',c)\right)\leq 0$$

$$\Rightarrow F\left(p(c,c'),p(c,c'),0,0,p(c,c'),p(c,c')\right)\leq 0. \quad (2.39)$$

If we assume $p(c,c_0) > 0$, because of the (F_2) property of F, we have

$$F\left(p(c,c'),p(c,c'),0,0,p(c,c'),p(c,c')\right)>0.$$

This contradicts with (2.39). Hence, $p(c,c')=0$, implying by (1.2) that $c=c'$.

Remark 2.3. We get corollaries from Theorem 2.2 by setting any of the following:

(i) $f=g$ and $S=T$,

(ii) $f=g$,
(iii) $S=T$,
(iv) $f=g=I$,
(v) $g = I$.

Example 2.4. Consider the function $F:\mathbb{R}^6 \to \mathbb{R}$ defined as

$F(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \lambda \max\{t_2/2, t_3, t_4, [t_5+t_6]/q\}$,

where $\lambda \in [0,1/3)$ and $q>1+2\lambda$. We show that F has the (F_{1A}) property. We note that F is non-increasing in the fifth variable. We also note that

$F(u_1,v_1,v_2,u_2,u_1+v_2,v_2-v_1) \le 0$ implies

$$u_1 - \lambda \max\{v_1/2, v_2, v_2, [u_1+2v_2-v_1]/q\} \le 0$$

$$\Rightarrow u_1 - \lambda \max\{v_2, u_2, [u_1+2v_2]/q\} \le 0. \tag{2.40}$$

If $\max\{v_2,u_2,[u_1+2v_2]/q\} = v_2$, then (2.40) becomes $u_1 \le \lambda v_2$.

If $\max\{v_2,v_2,[u_1+2v_2]/q\} = v_2$, then (2.40) becomes

$$\begin{aligned} u_1 &\le \lambda u_2 \\ &\le \lambda(u_1+v_2) \\ &\le \frac{\lambda}{1-\lambda} v_2 \\ &= hv_2, \text{ where } h = \frac{\lambda}{1-\lambda} < \frac{1}{2} \text{ for } 0 \le \lambda < \frac{1}{3}. \end{aligned}$$

Otherwise if $\max\{v_2,u_2,[u_1+2v_2]/q\} = [u_1+2v_2]/q$ then (2.40) implies

$$\begin{aligned} u_1 &\le \frac{\lambda}{q}(u_1+2v_2) \\ &\le \frac{2\lambda}{q-\lambda} v_2 \\ &\le \frac{2\lambda}{1+\lambda} v_2. \text{ because } 1+\lambda \le q-\lambda \\ &= hv_2, \text{ where } h = \frac{2\lambda}{1+\lambda} < \frac{1}{2} \text{ for } 0 \le \lambda < \frac{1}{3}. \end{aligned}$$

Thus in all cases, we have $u_1 \le hv_2, h \in [0,1/2)$.

The function also has the (F_2) property because for $u>0$ we have

$F(u,u,0,0,u,u) = u - \lambda \max\{u/2, 0, 0, 2u/q\}$

$$= u - \max\{u/2, 2u/q\}. \tag{2.41}$$

If $\max\{u/2, 2u/q\} = u/2$, then (2.41) becomes $u - (\lambda/2)u > 0$.

If however $\max\{u/2, 2u/q\} = 2u/q$, then (2.41) becomes $u - (2\lambda/q)u > 0$ because $q \geq 1 + 2\lambda$.

If (2.1) holds, this means

$$F\big(p(Sx,Ty), p(fx,gy), p(fx,Sx), p(gy,Ty), p(fx,Ty), p(gy,Sx) - p(gy,gy)\big) \leq 0$$

$$\Rightarrow p(Sx,Ty) - \lambda \max\{p(fx,gy)/2, p(fx,Sx), p(gy,Ty),$$

$$[p(fx,Ty) + p(gy,Sx) - p(gy,gy)]/q\} \leq 0$$

$$\Rightarrow p(Sx,Ty) \leq \lambda \max\{p(fx,gy)/2, p(fx,Sx), p(gy,Ty),$$

$$[p(fx,Ty) + p(gy,Sx) - p(gy,gy)]/q\}$$

$$\Rightarrow p(Sx,Ty) \leq \lambda \max\{p(fx,gy)/2, p(fx,Sx), p(gy,Ty),$$

$$[p(fx,Ty) + p(gy,Sx)]/q\}. \quad \text{... (2.42)}$$

The equation (2.42) describes an extension of the theorem proved by Imdad and Kumar [12] into partial metric spaces.

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