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$(\alpha, \beta) - L^p$ 2 -Norm Orthogonality and Characterizations of 2 - Inner Product Spaces

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ABSTRACT

In the present paper we have characterised $(\alpha, \beta) - L^p$ orthogonality in a 2-normed linear space. In some way the results proved in this paper generalize some of the similar characterization of generalized L^p- orthogonality derived earlier by Zheng Liu[8].

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INTRODUCTION

Recently there has been special interest to deal with certain analytic functional aspects in 2-normed spaces of finite or infinite dimensional type.

Usually orthogonality is dealt in inner product spaces but there is a concept like orthogonality in normal linear spaces ([2], [3], [5], [6],[7] and [8]). As has been noted earlier (for example see reference [7]) Birkhoff orthogonality plays a typical role in a normed linear space. In some analytic consideration also Birkhoff orthogonality is important.

In the present paper we have introduced $(\alpha, \beta) - L^p$ ⇳-orthogonality for a pair (x,z) and (y,z) in 2-normed spaces. We have also developed certain properties in the line of those given earlier by Liu [8] as was given in the normed spaces to be carried over in the setting of 2-normed space and 2-inner product spaces.

PRELIMINARIES AND NOTATIONS

DEFINITION 1. Let $p > 1$ be a fixed real number. If $(x, z) \in X \times X$, then we say that (x,z) is L^p-orthogonal and we denote $(x, z) \perp_{L^p} (y, z)$ provided

$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p$ is called left L^p orthogonality. In a similar way$(x, y) \perp_{L^p} (x, z)$ provided $\| x, y + z \|^p = \| x, y \|^p + \| x, z \|^p$ is called right L^p -orthogonality in 2-normed spaces.

DEFINITION 2. Let $p \geq 1$ and $\alpha, \beta \neq 1$ be fixed real numbers. If $(x, z) \in X \times X$ then (x,z) is 2-norm $(\alpha, \beta) - L^p-$ orthogonal to (y,z) denote by$(x, z) \perp_{L^p} (y, z)(\alpha, \beta)$ provided that

$$\| x + y, z \|^p + \| \alpha x + \beta y, z \|^p = \| \alpha x + y, z \|^p + \| x + \beta y, z \|^p$$

and $z \notin V(x, y)$ (where V(x,y) is the linear span of $x, y \in X$. Similarly we say (x,z) is $(\alpha, \beta) - L^p-$ orthogonal to (y,z) denoted by $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ provided that

$$\| x, y + z \|^p + \| x, \alpha y + \beta z \|^p = \| x, \alpha y + z \|^p + \| x, y + \beta z \|^p .$$

LEMMA 1. For all $(x, z), (y, z) \in X \times X, \alpha, \beta \neq 1, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only $((y, z) \perp_{L^p} (x, z))(\alpha, \beta)$.

The following theorem and corollary demonstrate that the concept of $(\alpha, \beta)-$ L^p- orthogonality is non-vacuous.

THEOREM 1. Let $p > 1$ and $\alpha, \beta \neq 1$ be a fixed real numbers. If $(x,z) \neq 0, (y,z) \in X \times X$ then there exists a real number such that $((x,z) \perp_{Lp} (ax+y,z))(\alpha,\beta)$.

PROOF. Set

$$f(t) = \| x+tx+y, z \|^p + \| ax+\beta(tx+y), z \|^p - \| ax+tx+y, z \|^p - \| x+\beta(tx+y), z \|^p .$$

Clearly, f is a continuous function on $-\infty < t < \infty$, and we have, for $t \neq 0$

$$f(t) = |t|^p[(\| x+\frac{1}{t}(x+y), z \|^p - \| x, z \|)^p - (\| \beta x+\frac{1}{t}(\alpha x+\beta y), z \|^p - \| \beta x, z \|)^p$$

$$-(\| x+\frac{1}{t}(\alpha x+y), z \|^p - \| x, z \|)^p - (\| \beta x+\frac{1}{t}(x+\beta y), z \|^p - \| \beta x, z \|)^p].$$

Then for $t \neq 0$

$$\frac{f(t)}{|t|^{p-1} sgnt} = \frac{\| x+\frac{1}{t}(x+y), z \|^p - \| x, z \|^p}{\frac{1}{t}} + \frac{\| \beta x+\frac{1}{t}(\alpha x+\beta y), z \| - \| \beta x, z \|^p}{\frac{1}{t}}$$

$$- \frac{\| x+\frac{1}{t}(\alpha x+y), z \|^2 - \| x, z \|^p}{\frac{1}{t}} - \frac{\| \beta x+\frac{1}{t}(x+\beta y), z \| - \| \beta x, z \|^p}{\frac{1}{t}},$$

and hence

$$\lim_{t \to -\infty}^{+} \frac{f(t)}{|t|^{p-1} sgnt} = p \| x, z \|^{p-1} J_{\pm}(x,z)(x+y) + p \| \beta x, z \|^{p-1} J_{\pm}(\beta x, z)(\alpha x+\beta y)$$

$$-p \| x \|^{p-1} J_{\pm}(x,z)(\alpha x+y) - p \| \beta x, z \|^{p-1} J_{\pm}(\beta x, z)(x+\beta y),$$

where $J_+((x,z)(y,z))$ and $J_-((x,z)(y,z))$ are respectively the right and left Gateaux derivative of the norm at (x,z), keeping second co-ordinate as fixed in the direction of (y,z). By James [5] we see that

$$J_{\pm}(x,z)(rx+sy) = r \| x, z \| + s J_{\pm}(x,z)(y,z)$$

for some $s \geq 0$ and r, therefore,

$$lim \frac{f(t)}{p-1} = p \| x, z \|^p (1 + \alpha\beta^{p-1} - \alpha - \beta^{p-1})$$

$$p \| x, z \|^p (1-\alpha)(1-\beta^{p-1}).$$

Thus for any fixed real number $\alpha, \beta \neq 1$ we have either $f(t) \to \infty$ as $t \to +\infty$ or $f(t) \to -\infty$. Hence there is a real number a such that $f(a) = 0$, which was to be proved.

COROLLARY 1. Let $p > 1$ and $\alpha, \beta \neq 1$ be the fixed real numbers. If $x \neq 0, z \neq 0, (y, z) \in X$, then there exist a real number a such that $((ax + yz) \perp_{L^p} (x, z)(\alpha, \beta))$.

PROOF. The result follows from Theorem 1 and Lemma 1.

Lemma 2. Let (x,z) or $(y, z) \in X \times X$ and $\alpha, \beta \neq 1$

(i) $\alpha, \beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((\alpha x, z) \perp_{L^p} (\beta y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,

(ii) if $\beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta))$ if and only if $((x, z) \perp_{L^p} \beta((y, z))(\alpha, \frac{1}{\beta})$,

(iii)if $\alpha \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((\alpha x, z) \perp_{L^p} (y, z))(\frac{1}{\alpha}, \beta)$.

Homogenity, symmetry and left and right additivity of $(\alpha, \beta) - L^p-$ orthogonality are defined in usual way, i.e. 2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous provided for all $x, y, z, \in X$ and real numbers $a, b, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies $((ax, z) \perp_{L^p} (by, z))(\alpha, \beta)$; 2-norm $(\alpha, \beta)-L^p-2$ norm orthogonality is symmetric provided for all $x, y, z \in X, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies $(y, z) \perp_{L^p} (x, z))(\alpha, \beta)$; 2 -norm $(\alpha, \beta) - L^p-$ orthogonality is left additive if and only if for all $x, y, w \in X, ((x, z) \perp_{L^p} (w, z))(\alpha, \beta)$ and $((y, z) \perp_{L^p} (w, z))(\alpha, \beta)$ implies $((x + y, z) \perp_{L^p} (w, z))(\alpha, \beta)$ and 2-norm $(\alpha, \beta) - L^p-$ orthogonality is right additive if and only if for all $(x, z), (y, z), (w, z) \in X \times X$, $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ and if $((y, z) \perp_{L^p} (w, z))(\alpha, \beta)$ imply $((x, z) \perp_{L^p} (y + w, z))(\alpha, \beta)$.

The following two corollaries are immediate consequences of the definition of homogeneity and Lemma 1 and Lemma 2.

COROLLARY 2. For all $\alpha, \beta \neq 1$,2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous if and only if 2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous.

COROLLARY 3. Suppose 2-norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous .

(i) If $\alpha, \beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_{L^p} (y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,

(ii) if $\beta \neq 0, ((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_{L^p} ((y, z))(\alpha, \frac{1}{\beta})$,

(iii)if $\alpha \neq 0, ((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$ if and only if $((x,z) \perp_{L^p} (y,z))(\frac{1}{\alpha},\beta)$. Now let us further study some consequences of homogeneity.

LEMMA 3. If $\alpha \neq -1$ and $(\alpha,\beta) - L^p - 2-$norm orthogonality is homogeneous, then $((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$ implies

$$\| x+y,z \|^p = (1-|\beta|^p) \| y,z \|^p + \| x+\beta y,z \|^p .$$

PROOF. From Corollary 3, it suffices to assume $|\alpha| < 1$. Suppose $((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$. Then $\| x+y,z \|^p + \| \alpha x+\beta y,z \|^p = \| \alpha x+y,z \|^p + \| x+\beta y,z \|^p$ keeping second co-ordinate as fixed. Since the result is immediate for $\alpha = 0$. We may assume that $\alpha \neq 0$. We are denoting the statement by $P(n)$ i.e.

$$P(n) : \| x+y,z \|^p + \| \alpha^n x+\beta y,z \|^p = \| \alpha^n x+y,z \|^p + \| x+\beta y,z \|^p .$$

Clearly $P(1)$ is true and if $P(n)$ is true for some positive integer n. Since 2-norm $(\alpha,\beta) - L^p -$ orthogonality is homogeneous, $((\alpha^n x,z) \perp_{L^p} (y,z))(\alpha,\beta)$, we have

$$\| \alpha^n x+y,z \|^p + \| \alpha^{n+1} x+\beta y,z \|^p = \| \alpha^{n+1} x+y,z \|^p + \| \alpha^n x+\beta y,z \|^p .$$

Adding this to $P(n)$ we obtain

$$\| x+y,z \|^p + \| \alpha^{n+1} x+\beta y,z \|^p = \| \alpha^{n+1} x+y,z \|^p + \| x+\beta y,z \|^p,$$

which is $P(n+1)$. Thus $P(n)$ is true for all positive integer n, but

$$\lim_{t \to \infty^+} \alpha^n = 0,$$

so in the limit, by continuity of the norm, we have

$$\| x+y,z \|^p + |\beta|^p \| y,z \|^p = \| y,z \|^p + \| x+\beta y,z \|^p,$$

and the conclusion of the lemma follows.

THEOREM 2. If $\alpha,\beta \neq -1$ and 2-norm $(\alpha,\beta) - L^p -$ orthogonality is homogeneous, then $((x,y) \perp_{L^p} (y,z))(\alpha,\beta)$ implies

$$\| x+y,z \|^p = \| x,z \|^p + \| y,z \|^p$$

i.e. 2 - norm $(\alpha, \beta) - L^p-$ orthogonality implies L_p orthogonality.

PROOF. By Corollary 6, we may assume $|\beta| < 1$. Suppose $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ and let $Q(n)$ denote the statement

$$Q(n) : \| x + y, z \|^p = (1 - |\beta^n|^p) \| y, z \|^p + \| x + \beta^n y, z \|^p .$$

The statement $Q(1)$ is Lemma 3. If we assume $Q(n)$ is true for some positive integer n, since $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ by homogeneity, we have by Lemma3,

$$\| x + \beta^n y, z \|^p = (1 - |\beta|^p) \| \beta^n y, z \|^p + \| x + \beta^{n+1} y, z \|^p$$

Substituting this in $Q(n)$ we obtain

$$\| x + y, z \|^p = (1 - |\beta^{n+1}|^p) \| y, z \|^p + \| x + \beta^{n+1} y, z \|^p$$

or $Q(n+1)$.

Hence $Q(n)$ holds for all positive integer n. Since $\beta^n \to 0$ as $n \to +\infty$ by taking limit in $Q(n)$ we obtain

$$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p .$$

THEOREM 3. If 2 - norm $(\alpha, \beta) - L^p-$ orthogonality is homogeneous, then $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies $\| x - y, z \| = \| x + y, z \|$ i.e. 2 - norm $(\alpha, \beta) - L^p-$ orthogonality implies 2 - norm isosceles orthogonality.

PROOF. If $(\alpha, \beta) = -1$ the result is follows. Otherwise by Lemma 1 we may assume without loss of generality that $\alpha \neq -1$. If $\beta = -1$ the result is immediate from Lemma 7. If $\beta \neq -1$, by Theorem 4, we have $\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p$. But by homogeneity $((x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ holds, so $\| x - y, z \|^p = \| x, z \|^p + \| -y, z \|^p$ and the result follows.

LEMMA 4. For all $\alpha, \beta \neq 1$, each one of the following:

(i) $(\alpha, \beta) - L^p$ 2 - norm orthogonality is symmetric and left additive,

(ii) $(\alpha, \beta) - L^p$ 2 - norm orthogonality is symmetric and right additive,

(iii) $(\alpha, \beta) - L^p$ 2 - norm orthogonality is left and right additive,

implies that $(\alpha, \beta) - L^p$ 2 - norm orthogonality is homogeneous.

PROOF. Suppose (i) holds and $((x,z) \perp_{L^p} (y,z))(\alpha,\beta)$, where x , y and z are arbitrary elements. Since the result is obvious for x = 0 or y = 0. We will assume $x \neq 0$ and $y \neq 0$. By Corollary 1, there exist a real number a such that $((ay-x,z) \perp_{L^p} (y,z))(\alpha,\beta)$. Left additive then gives $((ay,z) \perp_{L^p} (y,z))(\alpha,\beta)$ and hence $a = 0$. Thus $((-x,z) \perp_{L^p} (y,z))(\alpha,\beta)$, using left additive and symmetry, we find it now follows that $((nx,z) \perp_{L^p} (my,z))(\alpha,\beta)$ for all integers m and n, i.e.

$$\| nx+my,z \|^p + \| \alpha nx+\beta my,z \|^p = \| \alpha nx+my,z \|^p + \| nx+\beta my,z \|^p$$

or

$$\| x+\frac{m}{n}y,z \|^p + \| \alpha x+\beta\frac{m}{n}y,z \|^p = \| \alpha x+\frac{m}{n}y,z \|^p + \| x+\beta\frac{m}{n}y,z \|^p .$$

From the continuity of the norm it follows that

$$\| x+ky,z \|^p + \| \alpha x+\beta ky,z \|^p = \| \alpha x+ky,z \|^p + \| x+\beta ky,z \|^p$$

for all real numbers k or $((x,z) \perp_{L^p} (ky,z))(\alpha,\beta)$ for all k. So $(\alpha,\beta)-L^p$- 2 - norm orthogonality is homogeneous. By similar reasoning we can also get that (ii) and (iii) imply $(\alpha,\beta)-L^p-$ 2 - norm orthogonality is homogeneous. By similar reasoning, we can also get that (ii) and (iii) imply $(\alpha,\beta)-L^p-$ 2 - norm orthogonality is homogeneous.

The result may be summarized as follows:

THEOREM 4. Let $p > 1$ and $\alpha,\beta \neq 1$. The following are equivalent.

(i) $(\alpha,\beta)-L^p$ 2 - norm orthogonality is homogeneous,

(ii) $(\alpha,\beta)-L^p$ 2 - norm orthogonality is symmetric and left additive,

(iii)$(\alpha,\beta)-L^p$ 2 - norm orthogonality is symmetric and right additive,

(iv) $(\alpha,\beta)-L^p$ 2 - norm orthogonality is left and right additive.

Finally we give two characterizations of inner product spaces based on the relation between $(\alpha,\beta)-L^p$, 2 - norm orthogonality and Birkhoff 2 - norm orthogonality.

DEFINITION 3. If $(x,z),(y,z) \in X \times X$, we say (x,z) is Birkhoff orthogonal to (y,z), denoted $(x,z) \perp_\beta (y,z)$ provided $\| x+ky,z \| \geq \| x,z \|$ for all real numbers k.

THEOREM 5. Let $1 < p \leq 2$ and $0 < \alpha, \beta < 1$ be fixed real numbers. Then Birkhoff 2 - norm orthogonality implies $(\alpha, \beta) - L^p$ 2 - norm orthogonality in X if and only if X is an 2 - inner product space and p = 2.

PROOF. Let $(x, z) \perp_\beta (y, z)$. By assumption and homogeneity of Birkhoff 2 - norm orthogonality we get,

$$\| x + y, z \|^p = \| \alpha x + y, z \|^p + \| x + \beta y, z \|^p - \| \alpha x + \beta y, z \|^p$$

$$= (\| \alpha^2 x + y, z \|^p + \| \alpha x + \beta y, z \|^p - \| \alpha^2 x + \beta y, z \|^p)$$

$$+(\| \alpha x + \beta y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha x + \beta y, z \|^p) - \| \alpha x + \beta y, z \|^p$$

$$= (\| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p) - \| \alpha x + \beta^2 y, z \|^p + \| \alpha x + \beta y, z \|^p$$

$$= \| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p - \| \alpha x + \beta^2 y, z \|^p$$

$$+(\| \alpha^2 x + \beta y, z \|^p + \| \alpha x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta^2 y, z \|^p)$$

$$= \| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta^2 y, z \|^p .$$

Thus by induction we see that $(x, z) \perp_\beta (y, z)$. implies

$$\| x + y, z \|^p = \| \alpha^n x + y, z \|^p + \| x + \beta^n y, z \|^p - \| \alpha^n x + \beta^n y, z \|^p$$

for $n \geq 1$. In the limit this yields $(x, z) \perp_p (y, z)$. implies

$$\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p \qquad (A)$$

If for $p = 2$ then (A)yields

$$\| x + y, z \|^2 = \| x, z \|^2 + \| y, z \|^2 \qquad (B)$$

From the definition of 2 - inner product space, we have

$$\| x + y, z \|^2 = (x + y, x + y/z)$$

and

$$\| x, z \|^2 = (x, x/z)$$

$$\| y, z \|^2 = (y, y/z)$$

From (B) $(x+y, x/z)+(x+y, y/z) = (x, x/z)+(y, y/z)$
i.e. $(x, x/z)+y, x/z)+(x, y/z)(y, y/z)$
$= (x, x/z)+(y, y/z)+2(x, y/z) = 0 \Rightarrow (x, y/z) = 0$
Although in 1 - norm space the proof that identity (A) implies

$$\mu_p(X) = \sup_{(x,z)\perp_B(y,z)} \frac{(\| x, z \|^p + (\| y, z \|^p)^{\frac{1}{p}}}{(\| x+y, z \|^p} = 1$$

which in turn implies that X is an inner product space by the technique of Amir[1]. But one has to explore whether the same proof will work for $1 < p \leq 2$ in the context of 2 - norm spaces.
The other part is obvious.

THEOREM 6. Let $1 < p \leq 2$ and $0 < \alpha, \beta < 1$, be fixed real numbers. Then $(\alpha, \beta) - L^p$ 2 - norm orthogonality implies Birkhoff 2 - norm orthogonality in X if and only if X is an inner product space and p = 2.

PROOF. We first prove that if $(\alpha, \beta) - L^p$ 2 - norm orthogonality implies Birkhoff orthogonality then X is strict convex. If not then we can choose $x \neq y$ as extreme points of the units ball of X such that $\| x, z \| = \| y, z \|$
$= \| \frac{x+y}{z}, z \| = 1$. Then

$$\| \frac{x+y}{z}+y, z \|^p + \| \alpha\frac{x+y}{z}+\beta y, z \|^p \neq \| \alpha\frac{x+y}{z}+y, z \|^p + \| \frac{x+y}{z}+\beta y, z \|^p .$$

For otherwise $2^p + (\alpha+\beta)^p = (\alpha+1)^p + (\beta+1)^p$ which requires $\alpha = 1$ or $\beta = 1$, i.e. $(\frac{x+y}{z}, z)$ is not $(\alpha, \beta) - L^p$ 2 - norm orthogonal to (y,z). Without loss of generality we assume $\alpha \geq \beta$. By Theorem 2 we can choose $a \neq 0$ such that $((\frac{x+y}{z}, z) \perp_{L^p} (\frac{x+y}{z}+y, z))(\alpha+\beta)$. Hence $(\frac{x+y}{z}, z) \perp_{L^p} a\frac{x+y}{z}+y, z)$ i.e. $\| \frac{x+y}{z} + k(a\frac{x+y}{z}+y), z \| \geq \| \frac{x+y}{z}, z \| = 1$ for all real numbers k. Putting $k = -1/2$ yields $|\alpha| \leq 1$, and then $k = -\frac{1}{a+2}$ yields $|a+2| \leq 1$. Thus $a = -1$. But then $(\frac{x+y}{z}, z \perp_{L^p} a\frac{y-x}{z}, z)(\alpha, \beta)$ and then it gives

$$1+ \| \frac{\alpha-\beta}{z}x+\frac{\alpha+\beta}{z}y, z \|^p = \| \frac{\alpha-1}{z}x+\frac{\alpha+1}{z}y, z \|^p + \| \frac{1-\beta}{z}x+\frac{1+\beta}{z}y, z \|^p .$$

So we have

$$\alpha^p = \alpha^p \| \frac{\alpha-\beta}{2\alpha}x + \frac{\alpha+\beta}{2\alpha}y, z \|^p = \| \frac{\alpha-1}{z}x + \frac{\alpha+1}{z}y, z \|^p$$

and it follows $\| \frac{\alpha-1}{2\alpha}x + \frac{\alpha+1}{2\alpha}y, z \| = 1$.
Writing $(y,z) = \frac{1-\alpha}{1+\alpha}(x) + (1 - \frac{\alpha-1}{\alpha+1})(\frac{\alpha-1}{2\alpha}x + \frac{\alpha+1}{2\alpha}y)$.

We see that y is a convex combination of two points of the unit sphere which is false since y was taken to be an extrem point of the unit ball X. Thus X must be strictly convex.

Now we prove that if $(\alpha,\beta) - L^p$ 2 - norm orthogonality implies $(\alpha,\beta) - L^p$ 2 - norm orthogonality. If not, then there exists $(x,z)(y,z) \in X \times X$ such that $(x,z) \perp_\beta (y,z)$ and (x,z) is not $(\alpha,\beta) - L^p$ 2 - norm orthogonolity to (y,z). By Corollary 1 we can choose $b \neq 0$ such that $((by+x,z) \perp_{L^p} (y,z))(\alpha,\beta)$. But then $(by+x,z) \perp_\beta (y,z)$. Thus we have $(x,z) \perp_\beta (y,z)$ and $(by+x,z) \perp_\beta (y,z)$ which cotradicts the left uniqueness of 2 - norm Birkhoff orthogonality in strict convex space [5], hence 2 - norm Birkhoff orthogonality implies $(\alpha,\beta) - L^p$ 2 - norm orthogonality, which is sufficient for X to be an inner product space and $p = 2$ by Theorem 5.

The other part is also obvious.

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