ISSN: 0304-9787
The Aligarh Bulletin of Mathematics
Volume 36, Numbers 1-2 (2017) 145-152
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COMMON FIXED POINTS OF HYBRID PAIRS OF NON-SELF MAPPINGS SATISFYING AN IMPLICIT RELATION IN PARTIAL METRIC SPACES

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(Received)

Abstract

In this paper, we define a hybrid-type tangential property in the sense of Ahmed [8] in the setting of partial metric spaces. We obtain some results for coincidence and common fixed points of two hybrid pairs of non-self mappings satisfying an implicit relation due to Popa [19] under the tangential property in a partial metric space. Our results unify and generalize some existing ones in the literature.

1 Introduction and Preliminaries

Jungck and Rhoades [2] introduced the notion of compatible mappings for single-valued as well as multi-valued mappings in the study of fixed points. Singh

Keywords and phrases : Partial Metric Space, Implicit Relation, Hybrid-type Tangential Property, (E.A) *Property*.
AMS Subject Classification : 47H10, 54H25.

and Mishra [13] introduced the notion of (IT)-commuting mappings for hybrid mappings, and showed that (IT)-commuting mappings need not be weakly compatible. In 2002, Aamri and Moutawakil [7] introduced the *property* (E.A) and proved some metrical common fixed point theorems under strict contractive conditions. Kamran [16] extended the *property* (E.A) for a hybrid pair of mappings and generalized the notion of (IT)-commuting mappings for such pair of mappings. Liu, Wu, and Li [20] further extended the Kamran's notion for two hybrid pairs of mappings, and obtained some new fixed point results for such pairs. Kamran and Cakić [17] introduced a hybrid tangential property, and showed that the results in [20] can be proved under weaker forms than the *common property* (E.A).

On the other hand, Popa [19, 18] introduced the study of fixed points of mappings of a metric space satisfying an implicit relation, and showed that an implicit relation implies several contractive conditions. For an excellent discussions for both the Popa's approach and generalized versions of his results in [18], we refer the reader to [10]. The use of the combination of the *property* (E.A) or its generalizations and implicit relation are proving to be fruitful in common fixed point considerations [8, 9]. Motivated by the approach in [17], Ahmed [8] introduced a hybrid tangential condition, and obtained generalizations of some of the fixed point results in [9, 20].

Partial metric space is one of the most interesting generalizations of the notion of metric space which is such that the distance of a point from itself need not be zero [14]. Partial metric spaces were first introduced and studied by Matthews while studying denotational semantics of computer programming languages, showing that the essential tools of metric spaces like the Banach contraction principle can be generalized to partial metric spaces [14, 6]. Aydi, Abbas and Vetro [3] introduced and studied the notion of partial Hausdorff metric, and obtained the Nadler's fixed point theorem [11] in the setup of partial metric spaces. The tradition of establishing coincidence and common fixed point results for mappings under weaker commutativity conditions, and of using implicit relations for contraction conditions has been extended for partial metric spaces (see [15, 12, 1, 5] and references therein).

In this paper, we state and prove common fixed point theorems for two hybrid pairs of non-self mappings satisfying an implicit relation in partial metric spaces.

Let Y be a non-empty subset of X, and $T : Y \to 2^X$, where 2^X denotes the collection of all non-empty subsets of X, be a multi-valued mapping and $I : Y \to$

X be a single-valued mapping. Then a point $b \in Y$ is called a common fixed point of T and I if $b = Ib \in Tb$. A point $s \in Y$ is called a coincidence point of I and T if $Is \in Ts$. Let (X, d) be a metric space and $CB(X)$ denotes the collection of all non-empty bounded closed subsets of X, and $CL(X)$ denotes the collection of all non-empty closed subsets of X. For $A, B \in CL(X)$, define

$$H(A,B) = \begin{cases} \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}, & \text{if the maximum exists} \\ \infty, & \text{otherwise.} \end{cases}$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance from a point $x \in X$ to the set $A \in CL(X)$. Then H is called the generalized Pompeiu-Hausdorff metric on $CL(X)$ induced by the metric d. Notice that the pair $(CB(X), H)$ is a metric space.

The following definitions and facts will be frequently used in the sequel.

Definition 1. ([14]) *Let X be non-empty set. A partial metric space is a pair (X, p), where p is a function $p : X \times X \to [0, \infty)$, called the partial metric, such that for all $x, y, z \in X$:*

(P1) $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$; *and*

(P4) $p(x, y) + p(z, z) \leq p(x, z) + p(z, y)$.

Clearly, by (P1) - (P3), $p(x, y) = 0$ implies $x = y$. But, the converse is in general not true.

A classical example of partial metric spaces is the pair $([0, \infty), p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. For more examples of partial metric spaces, we refer the reader to [6, 14].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where
$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and ε is a real number.

Let (X, p) be a partial metric space, B any non-empty subset of the set X and x an element of the set X. It is shown in [4] that $x \in \bar{B}$, where $\bar{B}$ is the closure of B, if and only if $p(x, B) = p(x, x)$. Also, the set B is said to be closed in (X, p) if and

only if $B = \bar{B}$. A subset A is bounded in (X, p) if there exist $x_0 \in X$ and $r > 0$ such that $a \in B_p(x_0, r)$ for all $a \in A$; that is, $p(x_0, a) < p(a, a) + r, \quad \forall a \in A$.

Definition 2. ([14]) *A sequence $\{x_n\}$ in a partial metric space (X, p) is said to converge to some point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.*

Let (X, p) be a partial metric space and $CB^p(X)$ denotes the collection of all non-empty bounded and closed subsets of X and $CL^p(X)$ denotes the collection of all non-empty closed subsets of X. For $A, B \in CL^p(X)$ and $x \in X$, define

$$H_p(A, B) = \begin{cases} \max\{\delta_p(A, B), \delta_p(B, A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

where $p(x, A) = \inf\{p(x, a) : a \in A\}$, $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$ and $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$. Then, the mapping H_p is a generalized partial metric, called the generalized Pompeiu-Hausdorff partial metric, induced by the partial metric p. Moreover,it is shown in [3] that $(CB^p(X), H_p)$ is a partial metric space.

Definition 3. ([2]) *Let (X, d) be a metric space. Mappings $I : X \to X$, and $S : X \to CB(X)$ are said to be weakly compatible if $ISx = SIx$ whenever $Ix \in Sx$ for some $x \in X$.*

Definition 4. ([13]) *Let K be a non-empty subset of a metric space (X, d). Mappings $J : K \to X$ and $S : K \to CL(X)$ are (IT)-commuting at a point $x \in K$ if $JSx \subset SJx$ whenever $Sx \subset K$ and $Jx \in K$. J and S are (IT)-commuting if they are $(IT)-$commuting at each point $x \in K$.*

Definition 5. ([16]) *Let $S : X \to CB(X)$. The map $I : X \to X$ is said to be S-weakly commuting at $x \in X$ if $IIx \in SIx$.*

Definition 3 and Definition 5 extend to non-self mappings.

Definition 6. ([8]) *Let (X, d) be a metric space. Mappings $I : Y \subseteq X \to X$, and $S : Y \to CB(X)$ are said to be weakly compatible if $Ix \in Sx$ implies $ISx = SIx$ provided that $Ix \in Y$ and $Sx \subseteq Y$ for all coincidence points $x \in Y$ of I and S.*

Definition 7. ([8]) *Let $S : Y \subseteq X \to CB(X)$. The map $I : Y \subseteq X \to X$ is said to be S- weakly commuting at $x \in Y$ if $IIx \in SIx$ provided that $Ix \in Y$.*

Definition 8. ([7]) *Let (X, d) be a metric space. Mappings $I, J : X \to X$ are said to satisfy* property *(E.A) if there exists a sequence $\{x_n\} \subset X$ such that both $\{Ix_n\}$ and $\{Jx_n\}$ converge to t, for some $t \in X$.*

Definition 9. ([15]) *Let (X, p) be a partial metric space. Mappings $I, J : X \to X$ are said to satisfy* property *(E.A) if there exists a sequence $\{x_n\} \subset X$ such that both $\{Ix_n\}$ and $\{Jx_n\}$ converge to t, for some $t \in X$ and $p(t, t) = 0$.*

Definition 10. ([20]) *Let $I, J : X \to X$ and $S, T : X \to CB(X)$. The pairs (I, S) and (J, T) are said to satisfy the* common property *(E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X, some $t \in X$, and $A, B \in CB(X)$ such that $\lim_{n\to\infty} Sx_n = A$, $\lim_{n\to\infty} Ty_n = B$ and*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B.$$

Definition 11. ([8]) *Let (X, d) be a metric space, and $I, J : Y \subseteq X \to X$ and $T, S : Y \to CL(X)$ be maps. The hybrid pair (I, T) is said to be J- tangential at $t \in Y$ with respect to the map S if there exist two sequences $\{x_n\}$, $\{y_n\}$ in Y and $A \in CL(X)$ such that $\lim_{n\to\infty} Sy_n \in CL(X)$ and*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A = \lim_{n\to\infty} Tx_n.$$

Popa [19] intoduced the following implicit relation and proved some fixed point theorems for compatible mappings satisfying the relation. To describe the relation, let Ψ be the family of real lower semi-continuous functions $F(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(ψ_1) F is non-increasing in the variables t_3, t_4, t_5 and t_6,

(ψ_2) there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

- (ψ_{21}) $F(u, v, v, u, u + v, 0) \leq 0$ or
- (ψ_{22}) $F(u, v, u, v, 0, u + v) \leq 0$ we have $u \leq hv$, and

(ψ_3) $F(u, u, 0, 0, u, u) > 0$, $\forall u > 0$.

The following examples of such functions appear in [9, 19, 8].

Example 1. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1 - \frac{1}{2}t_2$.

Example 2. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1 - \lambda \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, *for* $\lambda \in (0, 1)$.

Example 3. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1 - h \max\left\{t_2, \frac{t_3 + t_4}{2}, \frac{t_5 + t_6}{2}\right\}$, *where* $h \in (0, 1)$.

Example 4. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{1 + t_3^2 + t_4^2}$, *where* $a > 0$, $b \geq 0$ *and* $a + b < 1$.

Example 5. *Define* $F(t_1, t_2, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$
as $F(t_1, t_2, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, *where* $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$, *and* $a + d < 1$.

In 2010, Ahmed [8] proved the following interesting common fixed point theorem for two pairs of non-self hybrid mappings satisfying the implicit relation described just above.

Theorem 1.1. *Let* I, J *be two maps from a subset* Y *of a metric space* (X, d) *into* X *and* S, T *be two maps from* Y *into* $CL(X)$. *Assume that*

(i) either the pair (I, T) *is* J*- tangential at* $t \in Y$ *with respect to the map* S *or the pair* (J, S) *is* I*- tangential at* $t \in Y$ *with respect to the map* T,

(ii) there exists a function $F \in \Psi$ *such that*
$F(H(Tx, Sy), d(Ix, Jy), d(Ix, Tx), d(Jy, Sy), d(Ix, Sy), d(Jy, Tx)) \leq 0$,
for all $x, y \in Y$.
Then,

(a) I *and* T *have a coincidence point* a *in* Y *provided* $I(Y)$ *is a closed subset of* X;

(b) J *and* S *have a coincidence point* b *in* Y *provided that* $J(Y)$ *is a closed subset of* X;

(c) I *and* T *have a common fixed point provided that* I *is* T*- weakly commuting at* a*,* $IIa = Ia$ *and* $Ia \in Y$*;*

(d) J *and* S *have a common fixed point provided that* J *is* S*- weakly commuting at* b*,* $JJb = Jb$ *and* $Jb \in Y$*;*

(e) I, J, S *and* T *have a common fixed point provided that both (c) and (d) are true.*

In this paper, we extend Theorem 1.1 to partial metric spaces.

2 Main Results

Now, we define the following notions for partial metric spaces.

Definition 12. *Let* (X, p) *be a partial metric space. Mappings* $I : Y \subseteq X \to X$*, and* $S : Y \to CL^p(X)$ *are said to be weakly compatible if for all* $x \in Y$ $Ix \in Sx$ *implies* $ISx = SIx$ *provided that* $Ix \in Y$ *and* $Sx \subseteq Y$*.*

Definition 13. *Let* $S : Y \subseteq X \to CL^p(X)$*. The map* $I : Y \subseteq X \to X$ *is said to be* S*- weakly commuting at* $x \in Y$ *if* $IIx \in SIx$ *provided that* $Ix \in Y$*.*

Notice that weakly compatibility leads to S- weakly commuting and the vice-versa is not true.

Definition 14. *Let* (X, p) *be a partial metric space, and* $I, J : X \to X$ *and* $S, T : X \to CL^p(X)$ *be mappings. The pairs* (I, S) *and* (J, T) *are said to satisfy the* common property *(E.A) if there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X*, some* $t \in X$*, and* $A, B \in CL^p(X)$ *such that* $\lim_{n\to\infty} Sx_n = A$*,* $\lim_{n\to\infty} Ty_n = B$ *and*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A \cap B \quad \textit{and} \quad p(t, t) = 0.$$

Definition 15. *Let* (X, p) *be a partial metric space. Let* $I, J : Y \subseteq X \to X$ *and* $T, S : Y \to CL^p(X)$ *be mappings. The hybrid pair* (I, T) *is said to be* J*-tangential at* $t \in Y$ *with respect to the map* S *if there exist two sequences* $\{x_n\}$*,* $\{y_n\} \subset Y$ *and* $A \in CL^p(X)$ *such that* $\lim_{n\to\infty} Sy_n \in CL^p(X)$*,*

$$\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Jy_n = t \in A = \lim_{n\to\infty} Tx_n \tag{2.1}$$

and $p(t, t) = 0$*.*

Remark 2.1. *The hybrid pairs (I,T) and (J,S) satisfy the* common property *(E. A) if and only if (I,T) is J-tangential with respect to the mapping S and (J,S) is I-tangential with respect to the mapping T, and the vice-versa is not true (see the following example).*

Example 6. *Let $X=[1,\infty)$ endowed with the partial metric $p(x,y)=\max\{x,y\}$ and $Y=[2,\infty)$. Define $I,J:Y\to X$ and $S,T:Y\to CL^p(X)$ by $Ix=3+\frac{1}{2}x$, $Jx=3+\frac{1}{5}x$, $Tx=[3,3+x]$, and $Sx=[1,3]$ for all $x\in Y$. Consider the sequences $\{x_n\}=\left\{1+\frac{1}{n}\right\}$ and $\{y_n\}=\left\{\frac{5}{2}+\frac{1}{n}\right\}$, $n\in\mathbb{N}$. Then we have the following:*

$$\lim_{n\to\infty} Ix_n=\lim_{n\to\infty} Jy_n=\frac{7}{2}\in[3,4]=\lim_{n\to\infty} Tx_n.$$

Hence, the hybrid pair(I,T) is J-Tagental with respect to S. It is clear that the hybrid pairs (I,T) and (J,S) do not satisfy the common property $(E.A)$, because that would imply that $\lim_{n\to\infty} x_n=\lim_{n\to\infty} y_n=0$.

Theorem 2.2. *Let (X,p) be a partial metric space. Let $I,J:Y\subseteq X\to X$ and $S,T:Y\to CL^p(X)$ be mappings such that*

(i) either the pair (I,T) is J- tangential at $t\in Y$ with respect to the mapping S or the pair (J,S) is I- tangential at $t\in Y$ with respect to the mapping T,

(ii) there exists a function $F\in\Psi$ such that

$$F(H_p(Tx,Sy),p(Ix,Jy),p(Ix,Tx),p(Jy,Sy),\\ p(Ix,Sy),p(Jy,Tx))\le 0 \tag{2.2}$$

for all $x,y\in Y$. Then,

(a) I and T have a coincidence point a in Y provided $I(Y)$ is a closed subset of X;

(b) J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X. Moreover,

(c) I and T have a common fixed point provided that I is T- weakly commuting at a, $IIa=Ia$ and $Ia\in Y$;

(d) J *and* S *have a common fixed point provided that* J *is* S*- weakly commuting at* b*,* $JJb = Jb$ *and* $Jb \in Y$*;*

(e) I, J, S *and* T *have a common fixed point provided that both (c) and (d) hold.*

Proof. Suppose that (I, T) is J-tangential with respect to the map S. Then, by Definition 15, there exist two sequences $\{x_n\}, \{y_n\} \subset Y$, $A \in CL^p(X)$ such that (2.1) holds. We claim that if $\lim_{n\to\infty} Sy_n = B$, then B equals A. Suppose $A \neq B$. Then $H_p(A, B) > 0$. Now from (2.2) we have

$$\begin{aligned} F(H_p(Tx_n, Sy_n), p(Ix_n, Jy_n), p(Ix_n, Tx_n), p(Jy_n, Tx_n) + H_p(Tx_n, Sy_n), \\ p(Ix_n, Tx_n) + H_p(Tx_n, Sy_n), p(Jy_n, Tx_n)) \leq \\ F(H_p(Tx_n, Sy_n), p(Ix_n, Jy_n), p(Ix_n, Tx_n), p(Jy_n, Tx_n), p(Ix_n, Tx_n), \\ p(Jy_n, Tx_n)) \leq 0 \end{aligned}$$

as $p(Jy_n, Sy_n) \leq p(Jy_n, Tx_n) + H_p(Tx_n, Sy_n)$,
$p(Ix_n, Sy_n) \leq p(Ix_n, Tx_n) + H_p(Tx_n, Sy_n)$, and F is non-increasing in the forth and fifth variables (see ψ_1).
Taking the limit as $n \to \infty$ gives
$F(H_p(A, B), 0, 0, H_p(A, B), H_p(A, B), 0) \leq 0$ as $p(t, t) = 0$.
Since $F \in \Psi$, from ψ_{21}, we have $H_p(A, B) \leq 0$. Thus, $A = B$.
(a) Next, we show that $Ia \in Ta$.
Suppose $I(Y)$ is closed. Then $\lim_{n\to\infty} Ix_n = t = Ia$ for some $a \in Y$.
Now, we claim that $Ta = A$. Suppose that $Ta \neq A$. Then $H_p(A, Ta) > 0$. From (2.2) we have

$$\begin{aligned} F(H_p(Ta, Sy_n), p(Ia, Jy_n), p(Ia, Ta), p(Jy_n, Sx_n), \\ p(Ia, Sy_n), p(Jy_n, Ta)) \leq 0. \end{aligned}$$

As $n \to \infty$, we get $F(H_p(Ta, A), 0, p(Ia, Ta), 0, 0, p(Ta, Ia)) \leq 0$.
Since $Ia \in A$ implies $p(Ta, Ia) \leq H_p(Ta, A)$, and F is non-increasing in the third and sixth variables (see ψ_1), by (2.2) we have:

$$\begin{aligned} F(H_p(Ta, A), 0, H_p(Ta, A), 0, 0, H_p(Ta, A)) \leq \\ F(H_p(Ta, A), 0, p(Ia, Ta), 0, 0, p(Ta, Ia)) \leq 0. \end{aligned}$$

So by ψ_{22}, we have $H_p(Ta, A) \leq 0$. Thus, $Ta = A$. Therefore, $a \in Y$ is a coincidence point of the mappings I and T i.e. $Ia \in Ta$.
(b) Similarly as above, one can prove that $Jb \in Sb$.
(c) Suppose that $IIa = Ia$ and the mapping I is T-weakly commuting at $a \in Y$ i.e. $IIa \in TIa$ (see Definition 13). Then $Ia = IIa \in TIa$ implies that Ia is a common fixed point of the pair (I, T).
(d) Similarly as in (c) above, one can show that Jb is a common fixed point of the pair (J, S).
(e) Since $Ia = Jb = t$, if both (c) and (d) hold then t is a common fixed point of the mappings I, J, T and S i.e. $t = It = Jt \in Tt \cap St$. □

By Remark 2.1, we have the following generalization of the results due to Kessy *et al.* [5] as Corollary to Theorem 2.2.

Corollary 2.3. *Let (X, p) be a partial metric space. Let $I, J : Y \subseteq X \to X$ and $S, T : Y \to CB^p(X)$ be mappings such that*

(i) (T, I) as well as (S, J) satisfy the property *(E.A),*

(ii) $\forall x, y \in X$,

$$H_p(Tx, Sy) \leq \lambda \max\{p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), \frac{1}{2}[p(Ix, Sy) + p(Jy, Tx)]\}$$

for all $x, y \in Y$ and $\lambda \in (0, 1)$,

(iii) $I(Y)$ and $J(Y)$ are complete subspaces of X. Then

(a) the pairs (T, I) and (S, J) have a coincidence point,

(b) the pairs (T, I) and (S, J) have a common fixed point provided that they are weakly compatible.

Corollary 2.4. *Let (X, p) be a partial metric space. Let $I, J, S, T : Y \subseteq X \to X$ be mappings such that*

(i) either the pair (I, T) is J- tangential at $t \in Y$ with respect to the mapping S or the pair (J, S) is I- tangential at $t \in Y$ with respect to the mapping T,

(ii) there exists a function $F \in \Psi$ such that

$$F(p(Tx, Sy), p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), p(Ix, Sy), p(Jy, Tx)) \leq 0$$

for all $x, y \in Y$. Then,

(a) I and T have a coincidence point a in Y provided $I(Y)$ is a closed subset of X;

(b) J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X. Moreover,

(c) I and T have a common fixed point provided that I is T- weakly commuting at a, $IIa = Ia$ and $Ia \in Y$;

(d) J and S have a common fixed point provided that J is S- weakly commuting at b, $JJb = Jb$ and $Jb \in Y$;

(e) I, J, S and T have a common fixed point provided that both (c) and (d) hold.

Corollary 2.5. *Let (X, p) be a partial metric space. Let $I, J : Y \subseteq X \to X$ and $S, T : Y \to CL^p(X)$ be mappings. Assume that*

(i) either the pair (I, T) is J- tangential at $t \in Y$ with respect to the mapping S or the pair (J, S) is I- tangential at $t \in Y$ with respect to the mapping T,

(ii) $\forall x, y \in X$,

$$H_p(Tx, Sy) \leq \lambda \max\{p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), \frac{1}{2}[p(Ix, Sy) + p(Jy, Tx)]\}$$

for all $x, y \in Y$ and $\lambda \in (0, 1)$. Then,

(a) I and T have a coincidence point a in Y provided $I(Y)$ is a closed subset of X;

(b) J and S have a coincidence point b in Y provided that $J(Y)$ is a closed subset of X. Moreover,

(c) I *and* T *have a common fixed point provided that* I *is* T*- weakly commuting at* a*,* $IIa = Ia$ *and* $Ia \in Y$*;*

(d) J *and* S *have a common fixed point provided that* J *is* S*- weakly commuting at* b*,* $JJb = Jb$ *and* $Jb \in Y$*;*

(e) I, J, S *and* T *have a common fixed point provided that both (c) and (d) hold.*

The following example illustrates the generality of Theorem 2.2 over Corollary 2.3.

Example 7. *Let* $Y = X = [0, \infty)$ *equipped with the partial metric*

$$p(x, y) = \begin{cases} |x - y|, & \text{if } 0 \leq x \leq 1 \\ \max\{x, y\}, & \text{if } x > 1. \end{cases}$$

.

Define $I, J : X \to X$ *and* $S, T : X \to CL^p(X)$ *by* $Tx = [x, \infty)$*,* $Sx = [x^2, \infty)$*,* $Ix = 2x$*,* $Jx = 2x^2$*. Clearly* $I(X), J(X) \in CL^p(X)$*.*

For all $x, y \in [0, 1]$ *we have :*

$$\begin{aligned} H_p(Tx, Sy) &= |x - y^2| \\ &= \frac{1}{2}|2x - 2y^2| \\ &= \frac{1}{2}|Ix - Jy| \\ &= \frac{1}{2}p(Ix, Jy). \end{aligned}$$

On the other hand, for all $x, y \in (1, \infty)$ *we have :*

$$\begin{aligned} H_p(Tx, Sy) &= \max\{x, y^2\} \\ &= y^2 \text{ if we suppose without loss of generality that } x \leq y \\ &= \frac{1}{2}2y^2 \\ &= \frac{1}{2}\max\{2x, 2y^2\} \\ &= \frac{1}{2}\max\{Ix, Jy\} \\ &= \frac{1}{2}p(Ix, Jy). \end{aligned}$$

Together, $H_p(Tx, Sy) = \frac{1}{2}p(Ix, Jy)$ *for all* $x, y \in X$.
Consider $F(t_1, t_2, \cdots, t_6) = t_1 - \frac{1}{2}t_2$ *as defined in Example 1. Then* $F(H_p(Tx, Sy), p(Ix, Jy), p(Ix, Tx), p(Jy, Sy), p(Ix, Sy), p(Jy, Tx)) = H_p(Tx, Sy) - \frac{1}{2}p(Ix, Jy) = 0$ *for all x, y in* X. *Moreover, it can be shown that* I *is* T*-weakly commuting at* 0 *and* J *is* S*-weakly commuting at* 0 *;* $II0 = I0 = 0$ *and* $JJ0 = J0 = 0$. *But,* $JS1 \neq SJ1$ *at* $J1 \in S1$. *Now, consider the sequences* $\{x_n\} = \left\{\frac{1}{n^2}\right\}$ *and* $\{y_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbb{N}$.
Then, it can be easily verified that the hybrid pair (I, T) *is* J*-Tangential with respect to* S*, and the hybrid pair* (J, S) *is* I*-Tangential with respect to* T*, and therefore* 0 *is a common fixed point.*

Remark 2.6. *Since the mappings* J *and* S *are not weakly compatible, then Corollary 2.3 is not applicable for this particular example.*

Conflict of Interest: The authors declare that there is no conflict of interests.

Acknowledgement: The authors are thankful to the learned referee for his valuable comments to improve the manuscript.

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