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COMMON FIXED POINT FOR A PAIR OF NON-SELF MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper we prove a common fixed point theorem for a pair of non-self mappings in partial metric spaces. We generalize a theorem by Imdad and Kumar which was proved for metric spaces. We also provide an illustrative example.

1. Introduction and Preliminaries

Frechet [3] introduced the concept of metric spaces in 1906. Since then researchers have been proving theorems that state the conditions for mappings in metric spaces to have fixed points. The theorem by Banach [2] is one of the most notable ones.

In 1994, as part of the study of denotational semantics of dataflow networks, Mathews [5] introduced a generalization of the metric space called the partial metric space. He also extended Banach's theorem to apply to partial metric spaces. Assad and Kirk [1] formulated fixed point theorems for non-self mappings in metrically convex metric spaces. Their result was extended by Rhoades [6] who proved the existence of fixed points in metric spaces for what he termed as generalized contractions. Imdad and Kumar [4] extended the theorem by Rhoades by demonstrating the existence of common fixed points for pairs of mappings obeying specified conditions in a metrically convex metric space.

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In this study we seek to generalize the theorem by Imdad and Kumar.
We will require the following preliminaries in developing our theorem.

Definition 1.1 ([5])**.** A partial metric space is a pair (X, p) consisting of a non-empty set X together with a function $p : X \times X \to \mathbb{R}$, called the partial metric, such that for all $x, y, z \in X$ we have the following properties:
P0: $0 \leq p(x, x) \leq p(x, y)$,
P1: $p(x, y) = p(x, x) = p(y, y)$ if and only if $x = y$,
P2: $p(x, y) = p(y, x)$ and
P3: $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

From P0 and P1 we have

$$p(x, y) = 0 \Rightarrow p(x, y) = p(x, x) = p(y, y) \Rightarrow x = y. \tag{1.1}$$

As an example, the pair $(\mathbb{R}^+, p)$, where and $p : X \times X \to \mathbb{R}^+$ is defined as $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, is a partial metric space.
Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2 ([5])**.**
(i) A sequence $\{x_n\}$ in a partial metric space (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Lemma 1.3 ([5])**.** *If p is a partial metric on X, then the mapping $p^s : X \times X \to [0, +\infty)$ given by*

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.2}$$

defines a metric on X.

In this paper we will denote p^s as the metric derived from the partial metric p.
We describe a metrically convex metric space and state some of its properties.

Definition 1.4 ([1])**.**
(i) A metric space (X, d) is said to be metrically convex if for all x, y in X with $x \neq y$, there exists a point z in $X, (x \neq z \neq y)$ such that

$$d(x, y) = d(x, z) + d(z, y).$$

(ii) Let (X, d) be a metrically convex metric space with $x, y \in X, x \neq y$. Then we define

$$\text{seg}[x, y] := \{z : d(x, y) = d(x, z) + d(z, y)\}.$$

As an example, the metric space $(\mathbb{R}^+, d)$, where the metric $d : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}^+$, is a convex metric space. This is because, for all $x, y, z \in \mathbb{R}^+$ with $x \neq y$ and $x < z < y$, we have $d(x, y) = d(x, z) + d(z, y)$.

We state some properties of metrically convex metric spaces.

Lemma 1.5 ([1])**.** *Let (X, d) be a metrically convex metric space with $x, y, z, w \in X$, $x \neq y$ and $z \in \text{seg}[x, y]$. Then*

$$d(z, w) \leq \max\{d(x, w), d(y, w)\}.$$

Lemma 1.6 ([1])**.** *Let K be a closed subset of set X, where (X, d) is a metrically convex metric space. If $x \in K$ and $y \in X\backslash K$, then there exists z in ∂K, (the boundary of K), such that $d(x, y) = d(x, z) + d(z, y)$.*

We define a metrically convex partial metric space.

Definition 1.7. A partial metric space (X, p) is said to be a metrically convex partial metric space if (X, p^s) is a metrically convex metric space, where p^s is the metric derived from the partial metric p.

As an example, the partial metric space $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, is metrically convex. This is because the metric space $(\mathbb{R}^+, p^s)$ is metrically convex, where $p^s(x, y) = |x - y|$ is the metric derived from the partial metric p.

We deduce the following lemma.

Lemma 1.8. *Let K be a closed subset of set X, where (X, p) is a metrically convex partial metric space. If $x \in K$ and $y \in X\backslash K$, then there exist a point z in ∂K, (the boundary of K), such that $z \in \text{seg}[x, y]$ and*

$$p(x, z) + p(z, y) = p(x, y) + p(z, z).$$

Proof. If (X, p) is a metrically convex partial metric space, then from Definition 1.7, (X, p^s) is a metrically convex metric space. This means that, from Lemma 1.6, if K is a closed subset of X, and we have $x \in K$ and $y \in X\backslash K$, then there is a point $z \in \partial K$ such that $p^s(x, y) = p^s(x, z) + p^s(z, y)$.

We now apply (1.2).

$$\begin{aligned}
p^s(x, y) &= p^s(x, z) + p^s(z, y), \\
\Rightarrow 2p(x, y) - p(x, x) - p(y, y) &= 2p(x, z) - p(x, x) - p(z, z) \\
&\quad + 2p(z, y) - p(z, z) - p(y, y), \\
\Rightarrow 2p(x, y) &= 2p(x, z) + 2p(z, y) - 2p(z, z), \\
\Rightarrow p(x, y) + p(z, z) &= p(x, z) + p(z, y), \\
\Rightarrow p(x, z) + p(z, y) &= p(x, y) + p(z, z).
\end{aligned}$$

Lemma 1.9. *Let (X, p) be a partial metric space and let p^s be the metric derived from the partial metric p. Then for all $a, b, c \in X$ we have*

$$p^s(b, c) \leq p^s(a, c) \Rightarrow p(b, c) \leq 2p(a, c).$$

Proof. Using (1.2) we have

$$\begin{aligned} p^s(b,c) &\leq p^s(a,c) \\ \Rightarrow 2p(b,c) - p(b,b) - p(c,c) &\leq 2p(a,c) - p(a,a) - p(c,c) \\ \Rightarrow 2p(b,c) &\leq 2p(a,c) - p(a,a) + p(b,b) \\ &\leq 2p(a,c) + p(b,b) \\ &\leq 2p(a,c) + p(b,c) \\ &\qquad (\text{as } p(b,b) \leq p(b,c) \text{ from P0}) \\ \Rightarrow p(b,c) &\leq 2p(a,c). \end{aligned}$$

Lemma 1.10. *Given a metrically convex partial metric space (X,p), and let $x,y,z \in X$, with $z \in \text{seg}[x,y]$. Then*

$$p(x,z) \leq p(x,y).$$

Proof. According to Lemma 1.8, in a metrically convex partial metric space (X,p), with $x,y,z \in X$ and $z \in \text{seg}[x,y]$, we have:

$$\begin{aligned} p(x,y) + p(z,z) &= p(x,z) + p(z,y), \\ \Rightarrow p(x,y) &= p(x,z) + p(z,y) - p(z,z). \end{aligned}$$

From Definition 1.1, P0 we have $p(z,y) - p(z,z) \geq 0$. Hence

$$p(x,y) \geq p(x,z) \Rightarrow p(x,z) \leq p(x,y).$$

Let $T : K \to X$ be a mapping, where $K \subseteq X$. We say that T is a *self mapping* if $K = X$, otherwise T is called a *non-self mapping*. If there is an element $x \in K$ such that $Tx = x$, we say that x is a *fixed point* of T in X.

Suppose we have two mappings $F, T : K \to X$, with $K \subseteq X$. Let there be $z \in K$ such that $Fz = Tz = w$. We say that z is a *coincidence point* of F and T in X. If $z = w$, then we call z a *common fixed point* of F and T in X.

Suppose we have two mappings $F, T : K \to X$ with $K \subseteq X$. We say F and T are *coincidentally commuting* if for all $z \in K$, $Fz = Tz$ implies $FTz = TFz$.

We now prove the following Lemma, which is modified from Theorem 1 of Assad and Kirk [1], as it is necessary for our work.

Lemma 1.11. *Consider a sequence $\{w_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ such that, for all $n \geq 2$ we have*

$$w_n \leq k \max\{w_{n-2}, w_{n-1}\}, k \in (0,1), \tag{1.3}$$

then

$$w_n \leq k^{n/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.4}$$

Proof. We prove it by the induction method. First we show that Lemma 1.11 holds for $n = 1$ and $n = 2$.

Note that $k \in (0, 1)$ implies $k^0 = 1$. When $n = 1$ we have

$$w_1 \leq k^0 \max\{w_0, w_1\} = k^{1/2}k^{-1/2} \max\{w_0, w_1\}. \tag{1.5}$$

We also note that $k \in (0, 1)$ implies $k < k^{1/2}$. Hence if $n = 2$, then (1.3) leads to

$$w_2 \leq k \max\{w_0, w_1\} \leq k^{1/2} \max\{w_0, w_1\} = k^{2/2}k^{-1/2} \max\{w_0, w_1\}. \tag{1.6}$$

We now show that, if Lemma 1.11 holds for $1 \leq n \leq j$ where $j \geq 2$, then it must be hold for $j + 1$. Hence we have from (1.3)

$$w_{j+1} \leq k \max\{w_{j-1}, w_j\}. \tag{1.7}$$

We consider two cases.

Case (i): Suppose $w_{j-1} \leq w_j$. Then (1.7) becomes

$$\begin{aligned} w_{j+1} &\leq kw_j \\ &\leq k \cdot k^{j/2}k^{-1/2} \max\{w_0, w_1\} \text{ from (1.4)} \\ &= k^{(j+2)/2}k^{-1/2} \max\{w_0, w_1\}. \end{aligned} \tag{1.8}$$

Case (ii): Suppose $w_{j-1} > w_j$. Then (1.7) becomes

$$\begin{aligned} w_{j+2} &\leq kw_{j-1} \\ &\leq k \cdot k^{(j-1)/2}k^{-1/2} \max\{w_0, w_1\} \text{ from (1.4)} \\ &= k^{(j+1)/2}k^{-1/2} \max\{w_0, w_1\}. \end{aligned} \tag{1.9}$$

We note that for $j \geq 2$ and $k \in (0, 1)$ we have $k^{(j+1)/2} > k^{(j+2)/2}$. Hence (1.8) and (1.9) imply that

$$w_{j+1} \leq k^{(j+1)/2}k^{-1/2} \max\{w_0, w_1\}.$$

2. Main Results

Rhoades [6] proved a theorem which set out sufficient conditions for a non- self mapping to have a unique fixed point in a metric space. Imdad and Kumar [4] extended this theorem to apply for a pair of non-self mappings. They introduced the following a generalized contractive condition to prove their theorem:

Definition 2.1 ([4])**.** Let (X, d) be a metric space, K a non-empty subset of X and $F, T : K \to X$ be mappings. If F and T satisfy the condition

$$\begin{aligned} d(Fx, Fy) \leq h \max\{&d(Tx, Ty)/2, d(Tx, Fx), d(Ty, Fy), \\ &[d(Tx, Fy) + d(Ty, Fx)]/q\} \end{aligned}$$

for all $x, y \in K, 0 < h < 1, q \geq 1 + 2h$, then F is called the generalized T-contractive mapping of K into X.

The theorem of Imdad and Kumar [4] is as follows:

Theorem 2.2 ([4]). *Let X be a Banach space, K a non-empty closed subset of X and $F, T : K \to X$ be such that F is a generalized T-contractive mapping of K into X. Furthermore let the following conditions hold:*
(i) $\partial K \subseteq TK, FK \cap K \subset TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$, and
(iii) TK is closed in X.
Then there exists a coincidence point z in K. Moreover if F and T are coincidentally commuting at z, then z remains a unique common fixed point of F and T.

In order to develop our main result we introduce the following definition in partial metric spaces.

Definition 2.3. Let (X, p) be a partial metric space, and $F, T : K \to X$ be mappings. If F and T satisfy the condition

$$p(Fx, Fy) \leq h \max\{p(Tx, Ty)/2, p(Tx, Fx), p(Ty, Fy), \\ [p(Tx, Fy) + p(Ty, Fx)]/q\}$$

for all $x, y \in K, 0 < 2h < 1, q \geq 1 + 2h$, then F is called the generalized T-contractive p-mapping of K into X.

In this paper, we seek to prove the following theorem.

Theorem 2.4. *Let (X, p) be a complete metrically convex partial metric space and K a closed subset of X with $\partial K \neq \emptyset$. Let $F, T : K \to X$ be such that F is a generalized T-contractive p-mapping of K into X. Furthermore let the following conditions hold:*
(i) $\partial K \subseteq TK, FK \cap K \subset TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$ and
(iii) TK is closed in X.
Then there exists a coincidence point z of F and Tin K. Moreover if F and T are coincidentally commuting at z, then z remains a unique common fixed point of F and T with $p(z, z) = 0$.

Proof. We generate the sequence $\{Tx_n\}$ in the following way. We commence with $w \in \partial K$. From assumption (i), we choose $x_0 \in K$ such that $Tx_0 = w$. We then find Fx_0. From assumption (ii), $Fx_0 \in K$. From assumption (i), we choose $x_1 \in K$ such that $Tx_1 = Fx_0$. We locate Fx_1. If $Fx_1 \in K$, then from assumption (i), we choose $x_2 \in K$ such that $Tx_2 = Fx_1$.

If however $Fx_1 \notin K$, then, as stated in Lemma 1.8, we can find $v \in \partial K$ such that $v \in \text{seg}\{Fx_0, Fx_1\}$. As $v \in \partial K$, from assumption (i), there is $x_2 \in K$ such that $Tx_2 = v$.

In general we generate the sequence $\{Tx_n\}, n \in \mathbb{N}$ as follows. We commence with an arbitrary $w \in \partial K$. From assumption (i), we choose $x_0 \in K$ such that $Tx_0 = w$. If $Fx_n \in K$, then from assumption (i), we choose $x_{n+1} \in K$ such that $Tx_{n+1} = Fx_n$.

Otherwise if $Fx_n \notin K$, then from Lemma 1.8, there is an element $v \in \partial K$ such that $v \in \text{seg}\{Fx_{n-1}, Fx_n\}$. From assumption (i), we choose $x_{n+1} \in K$ such that $Tx_{n+1} = v$.

The sequence $\{Tx_n\}$ can be partitioned into sets P and Q, where $P = \{Tx_i \in \{Tx_n\} : Tx_i = Fx_{i-1}, i \geq 1\}$ and $Q = \{Tx_i \in \{Tx_n\} : Tx_i \neq Fx_{i-1}\}$. We note that if $Tx_n \in Q$, then $Tx_n \in \partial K$ and $Tx_n \in \text{seg}\{Fx_{n-2}, Fx_{n-1}\}$.

We will show by contradiction that for $n \geq 1$, if $Tx_n \in Q$, then $Tx_{n-1} \in P$.

Suppose $Tx_{n-1} \in Q$. This means $Tx_{n-1} \in \partial K$. From (ii) we have $Fx_{n-1} \in K$, implying by assumption (i) that $Tx_n = Fx_{n-1}$, Hence $Tx_n \in P$, which is a contradiction.

We now consider the following cases.

Case 1: Suppose $(Tx_n, Tx_{n+1}) \in P \times P$.

From our assumption, for all $n \geq 1$, we have

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &= p(Fx_{n-1}, Fx_n) \\ &\leq h \max\{p(Tx_{n-1}, Tx_n)/2, p(Tx_{n-1}, Fx_{n-1}), p(Tx_n, Fx_n), \\ &\qquad [p(Tx_{n-1}, Fx_n) + p(Tx_n, Fx_{n-1})]/q\} \\ &= h \max\{p(Tx_{n-1}, Tx_n)/2, p(Tx_{n-1}, Tx_n), p(Tx_n, Tx_{n+1}), \\ &\qquad [p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Tx_n)]/q\}. \end{aligned}$$

From Definition 1.1 P3, we have

$$p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Tx_n) \leq p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1}),$$

hence

$$\begin{aligned} p(Tx_n, Tx_{n+1}) \leq h \max\{&p(Tx_{n-1}, Tx_n), \\ &[p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1})]/q\}. \end{aligned}$$

If $p(Tx_{n-1}, Tx_n) \geq [p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1})]/q$, then we have

$$p(Tx_n, Tx_{n+1}) \leq hp(Tx_{n-1}, Tx_n).$$

Otherwise if $p(Tx_{n-1}, Tx_n) < [p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1})]/q$, then we have

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq \frac{h}{q}[p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1})] \\ \Rightarrow p(Tx_n, Tx_{n+1}) &\leq \frac{h}{q-h} p(Tx_{n-1}, Tx_n). \end{aligned}$$

From the assumption, we have

$$q \geq 1 + 2h \Rightarrow q - h \geq 1 + h \Rightarrow \frac{1}{q-h} \leq 1 \Rightarrow \frac{h}{q-h} \leq h.$$

Hence we conclude that $p(Tx_n, Tx_{n+1}) \leq hp(Tx_{n-1}, Tx_n)$.

Thus for $(Tx_n, Tx_{n+1}) \in P \times P$, we have

$$p(Tx_n, Tx_{n+1}) \leq hp(Tx_{n-1}, Tx_n).$$

Case 2: Suppose $(Tx_n, Tx_{n+1}) \in P \times Q$.

As $Tx_n = Fx_{n-1}$ and $Tx_{n+1} \in \text{seg}[Fx_{n-1}, Fx_n]$, applying Lemma 1.10, we have

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &= p(Fx_{n-1}, Tx_{n+1}) \\ &\leq p(Fx_{n-1}, Fx_n) \\ &\leq hp(Tx_{n-1}, Tx_n). \end{aligned}$$

using a similar argument as for Case 1.

Case 3: Suppose for $n = 1, 2, \ldots$, we have $(Tx_n, Tx_{n+1}) \in Q \times P$ implying $Tx_{n-1} \in P$.

We now have $Tx_n \in \text{seg}[Fx_{n-2}, Fx_{n-1}]$ and $Tx_{n+1} = Fx_n$. Hence from Lemma 1.5, we have

$$\begin{aligned} p^s(Tx_n, Tx_{n+1}) &= p^s(Tx_n, Fx_n) \\ &\leq \max\{p^s(Fx_{n-2}, Fx_n), p^s(Fx_{n-1}, Fx_n)\}. \end{aligned}$$

(3.i) If $p^s(Fx_{n-2}, Fx_n) \leq p^s(Fx_{n-1}, Fx_n)$, then we have $p^s(Tx_n, Tx_{n+1}) = p^s(Tx_n, Fx_n) \leq p^s(Fx_{n-1}, Fx_n)$ which from Lemma 1.9 implies

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-1}, Fx_n) \\ &\leq 2h \max\{p(Tx_{n-1}, Tx_n)/2, p(Tx_{n-1}, Fx_{n-1}), \\ &\quad p(Tx_n, Fx_n), [p(Tx_{n-1}, Fx_n) + p(Tx_n, Fx_{n-1})]/q\} \\ &= 2h \max\{p(Tx_{n-1}, Tx_n)/2, p(Tx_{n-1}, Fx_{n-1}), \\ &\quad p(Tx_n, Tx_{n+1}), [p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Fx_{n-1})]/q\} \\ &= 2h \max\{p(Tx_{n-1}, Fx_{n-1}), \\ &\quad [p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Fx_{n-1})]/q\}. \end{aligned}$$

In the last step we have used Lemma 1.10.

Because $Tx_n \in \text{seg}\{Tx_{n-1}, Fx_{n-1}\}$, we have

$$p(Tx_{n-1}, Tx_n)/2 \leq p(Tx_{n-1}, Fx_{n-1}).$$

(3.i.1) Suppose $p(Tx_{n-1}, Fx_{n-1}) \geq [p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Fx_{n-1})]/q$, then we have $p(Tx_n, Tx_{n+1}) \leq 2hp(Tx_{n-1}, Fx_{n-1})$. Applying the argument as in Case 2, and the assumption that $2h < 1$, we have

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2hp(Tx_{n-1}, Fx_{n-1}) \\ &= 2hp(Fx_{n-2}, Fx_{n-1}) \\ &\leq 2h \cdot hp(Tx_{n-2}, Tx_{n-1}) \\ &< hp(Tx_{n-2}, Tx_{n-1}). \end{aligned}$$

(3.i.2) If it happens that
$p(Tx_{n-1}, Fx_{n-1}) \leq [p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Fx_{n-1})]/q$, then we have

$$p(Tx_n, Tx_{n+1}) \leq \frac{2h}{q}[p(Tx_{n-1}, Tx_{n+1}) + p(Tx_n, Fx_{n-1})].$$

Using P3 of Definition 1.1 on $p(Tx_{n-1}, Tx_{n+1})$ we get

$$\begin{aligned} p(Tx_n, Tx_{n+1}) \leq \frac{2h}{q}[&p(Tx_{n-1}, Tx_n) + p(Tx_n, Tx_{n+1}) - p(Tx_n, Tx_n) \\ &+ p(Tx_n, Fx_{n-1})]. \end{aligned} \tag{2.1}$$

From Lemma 1.8 we have

$$p(Tx_{n-1}, Tx_n) - p(Tx_n, Tx_n) + p(Tx_n, Fx_{n-1}) = p(Tx_{n-1}, Fx_{n-1}).$$

Because $q - 2h \geq 1$ as per the assumption, we have

$$2h \geq \frac{2h}{q - 2h}.$$

Hence (2.1) becomes

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq \frac{2h}{q}[p(Tx_n, Tx_{n+1}) + p(Tx_{n-1}, Fx_{n-1})] \\ \Rightarrow p(Tx_n, Tx_{n+1}) &\leq \left(\frac{2h}{q - 2h}\right) p(Tx_{n-1}, Fx_{n-1}) \\ &\leq 2hp(Tx_{n-1}, Fx_{n-1}) \\ &= 2hp(Fx_{n-2}, Fx_{n-1}) \\ &\leq 2h \cdot hp(Tx_{n-2}, Tx_{n-1}) \\ &< hp(Tx_{n-2}, Tx_{n-1}). \end{aligned}$$

The last two steps follow from Case 2, and the assumption that $2h < 1$.

(3.ii) Let us consider the situation where $(Tx_n, Tx_{n+1}) \in Q \times P$, with $p^s(Fx_{n-1}, Fx_n) < p^s(Fx_{n-2}, Fx_n)$. In this case we have

$$p^s(Tx_n, Tx_{n+1}) = p^s(Tx_n, Fx_n) \leq p^s(Fx_{n-2}, Fx_n)$$

which from Lemma 1.9 implies

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-2}, Fx_n) \\ &\leq 2h \max\{p(Tx_{n-2}, Tx_n)/2, p(Tx_{n-2}, Fx_{n-2}), \\ &\quad p(Tx_n, Fx_n), [p(Tx_{n-2}, Fx_n) + p(Fx_{n-2}, Tx_n)]/q\} \\ &= 2h \max\{p(Tx_{n-2}, Tx_n)/2, \\ &\quad p(Tx_{n-2}, Tx_{n-1}), p(Tx_n, Tx_{n+1}), \\ &\quad [p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Tx_n)]/q)\} \\ &= 2h \max\{p(Tx_{n-2}, Tx_n)/2, p(Tx_{n-2}, Tx_{n-1}), \\ &\quad [p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Tx_n)]/q)\}. \end{aligned} \tag{2.2}$$

From P3 of Definition 1.1, we have

$$\begin{aligned} p(Tx_{n-2}, Tx_n)/2 &\leq [p(Tx_{n-2}, Tx_{n-1}) + p(Tx_{n-1}, Tx_n) \\ &\quad - p(Tx_{n-1}, Tx_{n-1})]/2 \\ &\leq [p(Tx_{n-2}, Tx_{n-1}) + p(Tx_{n-1}, Tx_n)]/2 \\ &\leq \max\{p(Tx_{n-2}, Tx_{n-1}), p(Tx_{n-1}, Tx_n)\} \\ &= \max\{p(Tx_{n-2}, Tx_{n-1}), p(Fx_{n-2}, Tx_n)\} \\ &\leq \max\{p(Tx_{n-2}, Tx_{n-1}), p(Fx_{n-2}, Fx_{n-1})\}, \end{aligned}$$

because $p(Fx_{n-2}, Tx_n) \leq p(Fx_{n-2}, Fx_{n-1})$ from Lemma 1.10.

We may replace $p(Tx_{n-2}, Tx_n)/2$ on the right hand side of the inequality (2.2) with $\max\{p(Tx_{n-2}, Tx_{n-1}), p(Fx_{n-2}, Fx_{n-1})\}$. We get

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-2}, Fx_n) \\ &\leq 2h \max\{p(Tx_{n-2}, Tx_{n-1}), p(Fx_{n-2}, Fx_{n-1}), \\ &\quad p(Tx_{n-2}, Tx_{n-1}), \\ &\quad [p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Tx_n)]/q\} \\ &= 2h \max\{p(Tx_{n-2}, Tx_{n-1}), p(Fx_{n-2}, Fx_{n-1}), \\ &\quad [p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Tx_n)]/q\}. \end{aligned} \tag{2.3}$$

If the maximum on the right hand side of the inequality (2.3) is $p(Tx_{n-2}, Tx_{n-1})$, then we have

$$p(Tx_n, Tx_{n+1}) \leq 2hp(Tx_{n-2}, Tx_{n-1}).$$

If however the maximum is $p(Fx_{n-2}, Fx_{n-1})$ and because $2h < 1$, we have

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2hp(Fx_{n-2}, Fx_{n-1}) \\ &= 2h \cdot hp(Tx_{n-2}, Tx_{n-1}) \\ &< hp(Tx_{n-2}, Tx_{n-1}). \end{aligned}$$

In the situation where the maximum of the inequality (2.3) is $[p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Tx_n)]/q$, we have

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-2}, Fx_n) \\ &\leq \frac{2h}{q}[p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Tx_n)] \\ &\leq \frac{2h}{q}[p(Tx_{n-2}, Tx_{n+1}) + p(Tx_{n-1}, Fx_{n-1})] \\ &= \frac{2h}{q}[p(Tx_{n-2}, Tx_{n+1}) + p(Fx_{n-2}, Fx_{n-1})]. \end{aligned}$$

because from Lemma 1.10, we have

$$p(Tx_{n-1}, Tx_n) \leq p(Tx_{n-1}, Fx_{n-1}) = p(Fx_{n-2}, Fx_{n-1}).$$

Hence

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-2}, Fx_n) \\ &\leq \frac{2h}{q}[p(Tx_{n-2}, Tx_{n+1}) + hp(Tx_{n-2}, Tx_{n-1})]. \end{aligned}$$

We apply P3 of Definition 1.1 for $p(Tx_{n-2}, Tx_{n+1})$ and get

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-2}, Fx_n) \\ &\leq \frac{2h}{q}[p(Tx_{n-2}, Tx_{n-1}) + p(Tx_{n-1}, Tx_{n+1}) \\ &\qquad - p(Tx_{n-1}, Tx_{n-1}) + hp(Tx_{n-2}, Tx_{n-1})] \\ &\leq \frac{2h}{q}[p(Tx_{n-2}, Tx_{n-1}) + p(Tx_{n-1}, Tx_{n+1}) \\ &\qquad + hp(Tx_{n-2}, Tx_{n-1})]. \end{aligned}$$

But $p(Tx_{n-1}, Tx_{n+1}) = p(Fx_{n-2}, Fx_n)$. Hence

$$\begin{aligned} p(Tx_n, Tx_{n+1}) &\leq 2p(Fx_{n-2}, Fx_n) \\ &\leq \frac{2h}{q}[p(Tx_{n-2}, Tx_{n-1}) + p(Fx_{n-2}, Fx_n) \\ &\qquad + hp(Tx_{n-2}, Tx_{n-1})] \\ &= \frac{2h}{q}[(1+h)p(Tx_{n-2}, Tx_{n-1}) + p(Fx_{n-2}, Fx_n)] \\ \Rightarrow 2p(Fx_{n-2}, Fx_n) &\leq \frac{2h(1+h)}{q-h}p(Tx_{n-2}, Tx_{n-1}). \end{aligned}$$

As

$$q \geq 2h+1 \Rightarrow q-h \geq 1+h \Rightarrow \frac{1+h}{q-h} < 1,$$

we have

$$\frac{2h(1+h)}{q-h} < 2h.$$

Hence $p(Tx_n, Tx_{n+1}) \leq 2p(Fx_{n-2}, Fx_n) \leq 2hp(Tx_{n-2}, Tx_{n-1})$.

We have proved that in all the three cases, we have

$$p(Tx_n, Tx_{n+1}) \leq 2h \max\{p(Tx_{n-2}, Tx_{n-1}), p(Tx_{n-1}, Tx_n)\}. \tag{2.4}$$

We now apply Lemma 1.11 to (2.4), by letting $w_n = p(Tx_n, Tx_{n+1})$ and $k = 2h$. For $n \geq 1$, we get

$$p(Tx_n, Tx_{n+1}) \leq (2h)^{n/2}\delta \tag{2.5}$$

where

$$\delta = (2h)^{-1/2} \max\{p(Tx_0, Tx_1), p(Tx_1, Tx_2)\}.$$

Suppose $m, n \in \mathbb{N}, n > m$. Hence from P3 of Definition 1.1, we have

$$\begin{aligned} p(Tx_m, Tx_n) &\leq \sum_{i=m}^{n-1} p(Tx_i, Tx_{i+1}) - \sum_{i=m+1}^{n-1} p(Tx_i, Tx_i) \\ &\leq \sum_{i=m}^{n-1} p(Tx_i, Tx_{i+1}) \\ &\leq \sum_{i=m}^{n-1} (2h)^{i/2}\delta \\ &\leq \delta(2h)^{m/2}\frac{1-(2h)^{(n-m)/2}}{1-(2h)^{1/2}} \\ &\leq \delta(2h)^{m/2}\frac{1}{1-(2h)^{1/2}}. \end{aligned}$$

Taking $n, m \to +\infty$, we get

$$\lim_{n,m\to\infty} p(Tx_m, Tx_n) = 0 < \infty. \tag{2.6}$$

From Definition 1.2 (i), this means that $\{Tx_n\} \in K$ is a Cauchy sequence in (X, p). Since X is complete and TK closed, $\{Tx_n\}$ converges to a point $z \in TK$ that is

$$\lim_{m,n\to\infty} p(Tx_m, Tx_n) = \lim_{n\to\infty} p(z, Tx_n) = p(z, z) = 0. \tag{2.7}$$

We now show that z is a fixed point of both F and T.

Consider a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\{Tx_{n_k}\} \in P$. From the construction of proof, we have $Fx_{n_k-1} = Tx_{n_k} \in TK$ and $Tx_{n_k} \to z$, as $k \to +\infty$.

This means $Fx_{n_k-1} \to z \in K$ as $k \to +\infty$, because TK is closed. As $z \in TK$, there is a point $u \in K$ such that

$$Tu = z. \tag{2.8}$$

From (2.7), this means that $p(Tu, z) = p(z, Tu) = p(z, z) = 0$. Consider the following:

$$\begin{aligned} p(Fu, Fx_{n_k-1}) &\leq h\max\{p(Tu, Tx_{n_k-1})/2, p(Tu, Fu), p(Tx_{n_k-1}, Fx_{n_k-1}), \\ &\qquad [p(Tx_{n_k-1}, Fu) + p(Tu, Fx_{n_k-1})]/q\} \\ &= h\max\{p(z, Tx_{n_k-1})/2, p(z, Fu), p(Tx_{n_k-1}, Fx_{n_k-1}), \\ &\qquad [p(Tx_{n_k-1}, Fu) + p(z, Fx_{n_k-1})]/q\}. \end{aligned}$$

Taking limit $k \to \infty$ and noting that $p(z, z) = 0$ and $Tx_{n_k}, Fx_{n_k-1} \to z$, we get

$$\begin{aligned} p(Fu, z) &\leq h\max\{p(z, z)/2, p(z, Fu), p(z, z), \\ &\qquad [p(z, Fu) + p(z, z)]/q\} \\ \Rightarrow p(Fu, z) &\leq h\max\{0, p(z, Fu), 0, [p(z, Fu) + 0]/q\} \end{aligned}$$

$$\begin{aligned} &= hp(Fu, z) \\ \Rightarrow p(Fu, z) &= 0 \\ \Rightarrow Fu &= z, \text{ by equation (1.1).} \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we have $Fu = Tu = z$ which means u is a coincidence point of F and T.

Since F and T are coincidentally commuting, we have

$$Fu = Tu = z \text{ implies } FTu = TFu = Fz = Tz.$$

We claim that $Fz = z$, making z a common fixed point of F and T. We consider the following:

$$\begin{aligned} p(Fz, z) &= p(Fz, Fu) \\ &\leq h \max\{p(Tz, Tu)/2, p(Tz, Fz), p(Tu, Fu), \\ &\quad [p(Tz, Fu) + p(Tu, Fz)]/q\} \\ &= h \max\{p(Fz, z)/2, p(Fz, Fz), p(z, z), [p(Fz, z) + p(z, Fz)]/q\} \\ &= h \max\{p(Fz, z)/2, \frac{2}{q} p(Fz, z)\} \\ &= \max\{h/2, 2h/q, \}[p(Fz, z)]. \end{aligned}$$

Because both $2h/q$ and $1/2$ are less than 1, this means $p(Fz, z) = 0$, implying by equation (1.1)

$$Fz = z = Tz.$$

We have shown that z is a common fixed point of T and F. The uniqueness of the common fixed point follows easily. This completes the proof.

Corollary 2.5. *If we replace the partial metric p by a metric d, then Lemma 1.9 which states that*

$$p^s(b, c) \leq p^s(a, c) \Rightarrow p(b, c) < 2p(a, c),$$

becomes

$$p^s(b, c) \leq p^s(a, c) \Rightarrow d(b, c) < d(a, c).$$

With this property, we can replace the condition $0 \leq 2h < 1$ in Theorem 2.4 with $0 \leq h < 1$ which leads to the theorem proved by Imdad and Kumar [4] *as a corollary.*

If we set $T = I$, the identity mapping, in Theorem 2.4, we get the following corollary, which is the theorem by Rhoades [6] adapted for partial metric spaces.

Corollary 2.6. *Let (X, p) be a complete metrically convex partial metric space and K a closed subset of X with $\partial K \neq \emptyset$. Let $F : K \to X$ be such that*

$$p(Fx, Fy) \leq h \max\{p(x, y)/2, p(x, Fx), p(y, Fy), [p(x, Fy) + p(y, Fx)]/q\},$$

where $0 < 2h < 1, q \geq 2h + 1$. Furthermore let $x \in \partial K \Rightarrow Fx \in K$. Then there exists a unique fixed point z in K and $p(z, z) = 0$.

If we set $x = y$ in Theorem 2.4, we get the following corollary.

Corollary 2.7. *Let (X, p) be a complete metrically convex partial metric space and K a closed subset of X with $\partial K \neq \emptyset$. Let $F, T : K \to X$ be such that $p(Fx, Fx) \leq h \max\{p(Tx, Tx)/2, p(Tx, Fx)\}$. Furthermore let the following conditions hold:*
(i) $\partial K \subseteq TK, FK \cap K \subset TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$ and
(iii) TK is closed in X.
Then there exists a coincidence point z in K. Moreover if F and T are coincidentally commuting at z, then z remains a unique common fixed point of F and T with $p(z, z) = 0$.

Now we provide an example on the use of our result (Theorem 2.4).

Example 2.8. *Let (X, p) be a partial metric space with $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $K = [0, 1]$ and $F, T \to X$ be defined as*

$$Fx = 3x, Tx = 9x.$$

From the given information, $TK = [0, 9]$ is closed and $\partial K = \{0, 1\} \subset [0, 9] = TK$. We also have $FK = [0, 3]$ and $FK \cap K = [0, 1] \subset [0, 9] = TK$.
For $Tx \in \partial K = \{0, 1\}$, we have

$$x \in \{0, 1/9\} \Rightarrow Fx \in \{0, 1/3\} \subset [0, 1] = K.$$

We note that the pair F and T is coincidentally commuting as $F(0) = T(0)$ and $FT(0) = TF(0) = 0$.
Without loss of generality, suppose $x \leq y$ and $x, y \in K$. Then we have

$$\begin{aligned} p(Fx, Fy) &= \max\{Fx, Fy\} = \max\{3x, 3y\} = 3y \\ &= \frac{1}{3} \max\{9y, 3y\} \\ &= \frac{1}{3} \max\{Ty, Fy\} \\ &= \frac{1}{3} p(Ty, Fy) \\ &= hp(Ty, Fy) \end{aligned}$$

where $h = 1/3$. Thus, the condition $0 \leq 2h < 1$ in the theorem is obeyed.
Hence, for all $x, y \in K$, we have,

$$\begin{aligned} p(Fx, Fy) \leq h \max\{&p(Tx, Ty), p(Tx, Fx), p(Ty, Fy), \\ &[p(Tx, Fy) + p(Ty, Fx)]/q\}, \end{aligned}$$

where $0 \leq 2h < 1$. Thus all conditions of Theorem 2.4 are satisfied and 0 is the unique fixed point of F and T. Furthermore $p(0, 0) = 0$.

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