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# Fixed points for non-self mappings in multiplicative metric spaces

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**Abstract**

In this paper, we develop a theorem for a pair of non-self mappings. For this purpose, we define a multiplicative convex metric space and state the condition for a mapping in such space to have a fixed point. We explain the procedure of locating the fixed point. We also provide an illustrative example on the use of the theorem.

**Keywords**

Multiplicative metric space, non-self mapping, coincidence point, multiplicative convex set.

**AMS Subject Classification**

47H10, 54H25.

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## 1. Introduction and Preliminaries

A new type of non-Newton calculus, called multiplicative calculus, was developed by Grossman and Katz [5]. In this calculus the operations of subtraction and addition are replaced by division and multiplication. This calculus was more efficient than ordinary Newtonian calculus in addressing some types of differential equation problems (Bashirov et al. [3]). The calculus can also be utilized in biomedical image analysis (Florack and Assen [4]).

Inspired by multiplicative calculus, Özavşar and Çevikel [9] defined and developed the topological properties of the multiplicative metric space. They also obtained some fixed point results in complete multiplicative metric spaces.

Fixed points in multiplicative metric spaces have various applications such as solving multiplicative boundary value problems (Abbas et al. [1]) and determining the existence and uniqueness of solutions for a class of nonlinear integral equations (Jiang and Gu [7]).

Various researchers have developed fixed point theorems for self-mappings in multiplicative metric spaces. However studies proving fixed point theorem for non-self mappings in multiplicative metric spaces are very few. In this work, we will prove a fixed point theorem for pairs of non-self mappings in multiplicative metric spaces.

We introduce explanations which will be of use in this paper.

In this paper, $\mathbb{R}_+$ and $\mathbb{N}$ represent the set of positive real numbers and the set of natural numbers respectively. We also use the term MMS as an abbreviation of multiplicative metric space.

The following is the definition of a multiplicative metric space.

**Definition 1.1.** [1] Let $X$ be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is said to be a multiplicative metric on $X$ if for any $x, y, z \in X$, the following conditions hold:

(m1) $d(x,y) \geq 1$ and $d(x,y) = 1$ if and only if $x = y$;

(m2) $d(x,y) = d(y,x)$;

(m3) $d(x,y) \leq d(x,z) \cdot d(z,y)$.

The pair $(X,d)$ is called a multiplicative metric space.

Examples of a multiplicative metric spaces are stated here.

**Example 1.2.** [9] *Let* $d^* : (\mathbb{R}_+)^n \times (\mathbb{R}_+)^n \to [1,+\infty)$ *be defined as follows*

$$d^*(x,y) = \left|\frac{x_1}{y_1}\right|^* \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

*where* $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in (\mathbb{R}_+)^n$ *and* $|\cdot|^* : \mathbb{R}_+ \to [0,+\infty)$ *is defined as*

$$|a|^* = \begin{cases} a, & \text{if } a \geq 1 \\ \frac{1}{a}, & \text{if } a < 1. \end{cases}$$

*Then* $((\mathbb{R}_+)^n, d^*)$ *is a multiplicative metric space.*

The following example is modified from Özavşar and Çevikel [9].

**Example 1.3.** *Let* $a > 1$ *be a fixed number. Then* $d_a : \mathbb{R} \times \mathbb{R} \to [1,+\infty]$ *defined by*

$$d_a(x,y) = a^{|x-y|}$$

*holds the multiplicative metric conditions.*

**Definition 1.4.** [9] Let $(X,d)$ be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. Define the following set:

$$B_\varepsilon(x) := \{y \in X : d(x,y) < \varepsilon\},$$

which is called the multiplicative open ball of radius $\varepsilon$ with center $x$. Similarly, one can describe the multiplicative closed ball as follows:

$$\bar{B}_\varepsilon(x) := \{y \in X : d(x,y) \leq \varepsilon\}.$$

**Definition 1.5.** [9] Let $(X,d)$ be a multiplicative metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number $N$ such that if $n \geq N \Rightarrow x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative converging to $x$, denoted by $x_n \to_* x\,(n \to \infty)$.

**Lemma 1.6.** [9] *Let* $(X,d)$ *be a multiplicative metric space,* $\{x_n\}$ *be a sequence in* $X$ *and* $x \in X$. *Then* $x_n \to_* x$ *as* $n \to \infty$ *if and only if* $d(x_n,x) \to_* 1$ *as* $n \to \infty$.

**Lemma 1.7.** [9] *Let* $(X,d)$ *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in* $X$. *If the sequence* $\{x_n\}$ *is multiplicative convergent, then the multiplicative limit point is unique.*

**Definition 1.8.** [9] Let $(X,d)$ be a multiplicative metric space and $\{x_n\}$ be a sequence in $X$. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_m,x_n) < \varepsilon$ for all $m,n \geq N$.

**Lemma 1.9.** [9] *Let* $(X,d)$ *be a multiplicative metric space and* $\{x_n\}$ *be a sequence in* $X$. *Then* $\{x_n\}$ *is a multiplicative Cauchy sequence if and only if* $d(x_n,x_m) \to_* 1$ *as* $m,n \to \infty$.

**Definition 1.10.** [9] Let $(X,d)$ be a multiplicative metric space. A point $x \in X$ is said to be a multiplicative limit point of $S \subset X$ if and only if $(B_\varepsilon(x)\backslash\{x\}) \cap S \neq \emptyset$ for every $\varepsilon > 1$. The set of all multiplicative limit points of the set $S$ is denoted by $S'$.

**Definition 1.11.** [9] Let $(X,d)$ be a multiplicative metric space. A subset $S \subseteq X$ is called multiplicative closed in $(X,d)$ if $S$ contains all of its multiplicative limit points.

**Theorem 1.12.** [9] *Let* $(X,d)$ *be a multiplicative metric space and* $S \subset X$.

*(i) A point* $x \in X$ *belongs to* $\bar{S}$ *if and only if there exists a sequence* $(x_n)$ *in* $S$ *such that* $x_n \to_* x (n \to \infty)$.

*(ii) The set* $S$ *is multiplicative closed if and only if every multiplicative convergent sequence in* $S$ *has a multiplicative limit point that belongs to* $S$.

**Theorem 1.13.** [9] *Let* $(X,d)$ *be a complete multiplicative metric space and* $S \subseteq X$. *Then* $(S,d)$ *is complete if and only if* $S$ *is multiplicative closed.*

Using the common topological definition of the boundary of a set, we provide the following definition.

**Definition 1.14.** Let $(X,d)$ be a multiplicative metric space and $S \subseteq X$. The boundary of $S$, denoted by $\partial S$, is the set of points $x \in X$ such that every open ball $B_\varepsilon(x)$ contains at least one point of $S$ and at least one point not of $S$.

Inspired by the definition of a metrically convex metric space by Assad and Kirk [2], we define a multiplicative metrically convex MMS.

**Definition 1.15.** A complete MMS $(X,d)$ is said to be multiplicative metrically convex if $X$ has the property that for each $x,y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$, such that $d(x,y) = d(x,z) \cdot d(z,y)$.

If $(X,d)$ is a metrically convex metric space, and $x,y \in X$, we term

$$\text{seg}[x,y] := \{z \in X : d(x,y) = d(x,z) \cdot d(z,y)\}. \qquad (1.1)$$

We state an example of a multiplicative convex MMS.

**Example 1.16.** *Let* $d^* : \mathbb{R}_+ \times \mathbb{R}_+ \to [1,+\infty]$ *be defined as*

$$d^*(x,y) = \left|\frac{x}{y}\right|^*,$$

*where*

$$|a|^* = \begin{cases} a, & \text{if } a \geq 1 \\ \dfrac{1}{a}, & \text{if } a < 1 \end{cases}.$$

*Then* $(\mathbb{R}_+, d^*)$ *is a multiplicative convex MMS. Without loss of generality, let* $x,y \in \mathbb{R}_+$ *be such that* $x < y$. *Then for all* $z$ *such that* $x < z < y$, *we have* $z \in \text{seg}[x,y]$.



**Lemma 1.17.** *Let $C$ be a multiplicative closed subset of the complete and multiplicative convex MMS $(X,d)$. If $x \in C$ and $y \notin C$, then there exists a point $z \in \partial C$ (the boundary of $C$) such that $z \in \text{seg}[x,y]$.*

*Proof.* From Definition 1.15, for $x,y \in X$ and $z \in seg[x,y]$, we have

$$\begin{aligned} & d(x,z)\cdot d(z,y) = d(x,y) \\ & \quad \Rightarrow d(x,z) \le d(x,y), \text{ because } d(y,z) \ge 1 \\ & \quad \Rightarrow d(x,z) = d(x,y)^t \text{ for some } t \in (0,1). \end{aligned}$$

We can write $z$ as a function of $t$ as

$$z(t) = \{z \in \text{seg}[x,y] : d(x,z) = d(x,y)^t\}.$$

When $t \to 0$, we have

$$d(x,z) \to 1 \Rightarrow z \to x, \text{ by (m1) of definition 1.1.}$$

When $t \to 1$, we have

$$d(x,z) \to d(x,y) \Rightarrow z \to y.$$

The exponential function is a continuous function. Thus $z$ traces a continuous curve from $x$ to $y$.

If $x \in C$ and $y \in X \backslash C$, the continuous curve traced by $z$ will intersect the boundary of $C$ on at least one point. This proves the lemma. □

We introduce the following lemma.

**Lemma 1.18.** *Let $(X,d)$ be a multiplicative metrically convex MMS. Let $x,y \in X$. If $z \in \text{seg}[x,y]$ then*

*(i)* $d(x,z) \le d(x,y)$ *and*

*(ii)* $d(y,z) \le d(x,y)$.

*Proof.* From Definition 1.15, we have

$$\begin{aligned} & d(x,z)\cdot d(z,y) = d(x,y) \\ & \Rightarrow d(x,z) = \frac{d(x,y)}{d(z,y)} \le d(x,y) \text{ because } d(z,y) \ge 1. \end{aligned}$$

Similarly,

$$d(y,z) = \frac{d(x,y)}{d(x,z)} \le d(x,y) \text{ because } d(x,z) \ge 1.$$

□

In this paper, the following lemma by Rugumisa and Kumar [10] is used.

**Lemma 1.19.** [10] *Consider a sequence $\{w_n\}_{n\in\mathbb{N}} \in \mathbb{R}_+$ such that for all*
*$n \ge 2$, we have*

$$w_n \le k \max\{w_{n-2}, w_{n-1}\}, k \in (0,1),$$

*then*

$$w_n \le k^{n/2} k^{-1/2} \max\{w_0, w_1\}.$$

An element $z \in C$ is called a *coincidence point* of mappings $S : C \to X$ and $T : C \to X$ if $Sz = Tz$. Furthermore, if at a coincidence point $z$ we have $Sz = Tz = z$, then $z$ is called a *common fixed point* of $S$ and $T$.

Two mappings $S$ and $T$ are said to be *coincidentally commuting* if $STz = TSz$ at all points where $z$ is a coincidence point.

Khan and Imdad [8] proved the following fixed point theorem for self mappings in MMS.

**Theorem 1.20.** [8] *Let $A,B,I$ and $J$ be self-mappings of a multiplicative metric space $(X,d)$ satisfying $A(X) \subset J(X)$, $B(X) \subset I(X)$ and*

$$\begin{aligned} d(Ax,By) \le & \big[\max\{d(Ix,Jy), d(Ix,Ax), d(By,Jy), \\ & d(Ax,Jy), d(Ix,By)\}\big]^{\lambda}, \\ \lambda \in (0,\tfrac{1}{2}) & \text{ for all } x,y \in X. \end{aligned}$$

*If one of $A(X),B(X),I(X),J(X)$ is a complete subspace of X, then the*
*following conclusions hold*
*(i) (A,I) has coincidence point,*
*(ii) (B,J) has coincidence point.*
*Further if the pairs $(A,I)$ and $(B,J)$ are coincidently commuting, then $A,B,I$ and $J$ have a unique common fixed point.*

The aim of this paper is to modify Theorem 1.20 so that it applies to a pair of non-self mappings in multiplicative metric spaces.

## 2. Main Results

We prove the following theorem.

**Theorem 2.1.** *Let (X,d) be a multiplicative metric space which is complete and multiplicative metrically convex. Let the mappings $S,T : C \to X$, where $C$ is a multiplicative closed subset of $X$ with a non-empty boundary $\partial C$, obey the following conditions:*

*(i)* $d(Sx,Sy) \le [\max\{d(Tx,Ty), d(Tx,Sx), d(Ty,Sy), d(Tx,Sy), (d(Ty,Sx)\}]^{\lambda}$, *where* $\lambda \in \left(0,\frac{1}{3}\right)$,

*(ii)* $\partial C \subseteq TC, SC \cap C \subset TC$,

*(iii)* $Tx \in \partial C$ *implies* $Sx \in C$,

*(iv)* $TC$ *is multiplicative closed.*

*Then $S$ and $T$ have a coincidence point in $C$. Moreover if $S$ and $T$ are coincidentally commuting, then they have a unique common fixed point in $C$.*

*Proof.* We generate the sequences $\{Sx_n\}$ and $\{Tx_n\}$ in the following way. We commence with $w \in \partial C$. From assumption (ii), we can choose $x_0 \in C$ such that $Tx_0 = w$. We then find



$Sx_0$. From assumption (iii), $Sx_0 \in C$. From (i), we can choose $x_1 \in C$ such that $Tx_1 = Sx_0$. We then find $Sx_1$.

The other elements in the sequences $\{Sx_n\}$ and $\{Tx_n\}$ are generated as follows:

If $Sx_n \in C$, then by (i), we can choose $x_{n+1} \in C$ such that $Tx_{n+1} = Sx_n$. We then find $Sx_{n+1}$.

Otherwise if $Sx_n \notin C$, then by Lemma 1.17, there is an element $v \in \partial C$ such that $v \in \text{seg}\{Sx_{n-1}, Sx_n\}$. From assumption (ii), we choose $x_{n+1} \in C$ such that $Tx_{n+1} = v$. We then find $Sx_{n+1}$.

We partition the sequence $\{Tx_n\}$ into two sets $P$ and $Q$ where $P = \{Tx_n \in \{Tx_n\} : Tx_n = Sx_{n-1}, i \geq 1\}$ and $Q = \{Tx_n \in \{Tx_n\} : Tx_n \neq Sx_{n-1}\}$.

From the construction of sequence, we note that when $Tx_n \in Q$, then $Tx_n \in \text{seg}[Sx_{n-1}, Sx_n]$ and $Tx_n \in \partial C$.

We consider the following cases.

**Case 1.** Consider $(Tx_n, Tx_{n+1}) \in P \times P$, for $n \geq 1$. This implies $(Tx_n, Tx_{n+1}) = (Sx_{n-1}, Sx_n)$.

From (i) in the assumption, we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(Sx_{n-1}, Sx_n) \\
&\leq [\max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Sx_{n-1}), \\
&\qquad d(Tx_n, Sx_n), d(Tx_{n-1}, Sx_n), d(Tx_n, Sx_{n-1})\}]^{\lambda} \\
&= [\max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n), \\
&\quad d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_{n+1}), d(Tx_n, Tx_n)\}]^{\lambda} \\
&= [\max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_{n+1})\}]^{\lambda}, \\
&\qquad \text{by (m1) of Definition 1.1,} \\
&\leq [\max\{d(Tx_{n-1}, Tx_n), \\
&\qquad d(Tx_{n-1}, Tx_n) \cdot d(Tx_n, Tx_{n+1})\}]^{\lambda}, \text{ by (m3),} \\
&= [d(Tx_{n-1}, Tx_n) \cdot d(Tx_n, Tx_{n+1})]^{\lambda}, \text{ by (m1).}
\end{aligned}$$

The expression above implies

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq \left[d(Tx_{n-1}, Tx_n)\right]^{\frac{\lambda}{1-\lambda}} \\
&= \left[d(Tx_{n-1}, Tx_n)\right]^{k}, \text{ where } k = \frac{\lambda}{1-\lambda}.
\end{aligned} \tag{2.1}$$

**Case 2.** Consider $(Tx_n, Tx_{n+1}) \in P \times Q, n \geq 1$. From the construction of sequence, this means $Tx_n = Sx_{n-1}$ and $Tx_{n+1} \in \text{seg}[Sx_{n-1}, Sx_n]$.

From (i), we have

$$\begin{aligned}
d(Sx_{n-1}, Sx_n) &\leq [\max\{d(Tx_{n-1}, Tx_n), \\
&\qquad d(Tx_{n-1}, Sx_{n-1}), d(Tx_n, Sx_n), \\
&\qquad d(Tx_{n-1}, Sx_n), d(Tx_n, Sx_{n-1})\}]^{\lambda} \\
&= [\max\{d(Tx_{n-1}, Sx_{n-1}), d(Tx_{n-1}, Sx_{n-1}), \\
&\qquad d(Sx_{n-1}, Sx_n), d(Tx_{n-1}, Sx_n), d(Tx_n, Tx_n)\}]^{\lambda} \\
&= [\max\{d(Tx_{n-1}, Sx_{n-1}), d(Tx_{n-1}, Sx_n)\}]^{\lambda}, \\
&\qquad \text{using (m1) of Definition 1.1,} \\
&\leq [\max\{d(Tx_{n-1}, Sx_{n-1}), \\
&\qquad d(Tx_{n-1}, Sx_{n-1}) \cdot d(Sx_{n-1}, Sx_n)\}]^{\lambda}, \text{ by (m3),} \\
&= [d(Tx_{n-1}, Sx_{n-1}) \cdot d(Sx_{n-1}, Sx_n)]^{\lambda} \\
&\leq [d(Tx_{n-1}, Sx_{n-1})]^{\frac{\lambda}{1-\lambda}} \\
&= [d(Tx_{n-1}, Tx_n)]^{k} \text{ where } k = \frac{\lambda}{1-\lambda}.
\end{aligned} \tag{2.2}$$

Because $Tx_{n+1} \in \text{seg}[Sx_{n-1}, Sx_n]$, we apply Lemma 1.18 to (2.2) and get

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(Sx_{n-1}, Tx_{n+1}) \\
&\leq d(Sx_{n-1}, Sx_n) \\
\Rightarrow d(Tx_n, Tx_{n+1}) &\leq [d(Tx_{n-1}, Tx_n]^{k}.
\end{aligned} \tag{2.3}$$

**Case 3.** Consider $(Tx_n, Tx_{n+1}) \in Q \times P$. In this scenario, we have $Tx_{n+1} = Sx_n$. We also have $Tx_n \in \text{seg}[Sx_{n-2}, Sx_{n-1}]$.

**Remark 1.** We claim that $Tx_n \in Q$ implies $Tx_{n-1} \in P$. We prove this claim by contradiction.

Suppose $Tx_{n-1} \in Q$. This implies that $Tx_{n-1} \in \partial C$. From (iii) in the assumption, this means $Tx_n = Sx_{n-1} \in C$. This implies $Tx_n \in P$ which is a contradiction. Hence $Tx_n \in Q$ implies $Tx_{n-1} \in P$.

We note that $Tx_{n-1} \in P$ implies $Tx_{n-1} = Sx_{n-2}$.

Applying (m3) of Definition 1.1, we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(Tx_n, Sx_n) \\
&\leq d(Tx_n, Sx_{n-1}) \cdot d(Sx_{n-1}, Sx_n).
\end{aligned} \tag{2.4}$$

*Subcase 3.1.* Suppose $d(Tx_n, Sx_{n-1}) \leq d(Sx_{n-1}, Sx_n)$. Then (2.4) leads to

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq d(Sx_{n-1}, Sx_n) \cdot d(Sx_{n-1}, Sx_n) \\
&= [d(Sx_{n-1}, Sx_n)]^2.
\end{aligned} \tag{2.5}$$

We consider $d(Sx_{n-1}, Sx_n)$. Using the argument in Case 2, we get

$$d(Sx_{n-1}, Sx_n) \leq [d(Tx_{n-1}, Tx_n)]^{k}, \text{ where } k = \frac{\lambda}{1-\lambda}. \tag{2.6}$$

Applying (2.6) to (2.5), we get

$$d(Tx_n, Tx_{n+1}) \leq [d(Tx_{n-1}, Tx_n)]^{2k}. \tag{2.7}$$



*Subcase 3.2.* Suppose $d(Tx_n, Sx_{n-1}) > d(Sx_{n-1}, Sx_n)$. Then (2.4) becomes

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\le d(Tx_n, Sx_{n-1}) \cdot d(Tx_n, Sx_{n-1}) \\ &= [d(Tx_n, Sx_{n-1})]^2 \\ &\le [d(Sx_{n-2}, Sx_{n-1})]^2. \text{ by Lemma 1.18.} \end{aligned} \tag{2.8}$$

Let us consider $d(Sx_{n-2}, Sx_{n-1})$. Using (i) in the assumption, we get

$$\begin{aligned} d(Sx_{n-2}, Sx_{n-1}) &\le [\max\{d(Tx_{n-2}, Tx_{n-1}), \\ &\quad d(Tx_{n-2}, Sx_{n-2}), d(Tx_{n-1}, Sx_{n-1}), \\ &\quad d(Tx_{n-2}, Sx_{n-1}), d(Tx_{n-1}, Sx_{n-2})\}]^\lambda \\ &= [\max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-2}, Tx_{n-1}), \\ &\qquad d(Sx_{n-2}, Sx_{n-1}), d(Tx_{n-2}, Sx_{n-1}), \\ &\qquad d(Tx_{n-1}, Tx_{n-1})\}]^\lambda \\ &= [\max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-2}, Sx_{n-1})\}]^\lambda \\ &\qquad \text{by (m1) of Definition 1.1,} \\ &\le [\max\{d(Tx_{n-2}, Tx_{n-1}), \\ &\qquad d(Tx_{n-2}, Tx_{n-1}) \cdot d(Tx_{n-1}, Sx_{n-1})\}]^\lambda, \\ &\qquad \text{by (m3)} \\ &= [d(Tx_{n-2}, Tx_{n-1}) \cdot d(Tx_{n-1}, Sx_{n-1})\}]^\lambda \text{ by (m1)} \\ &= [d(Tx_{n-2}, Tx_{n-1}) \cdot d(Sx_{n-2}, Sx_{n-1})\}]^\lambda \\ &\le [d(Tx_{n-2}, Tx_{n-1})]^k, \text{ where } k = \frac{\lambda}{1-\lambda}. \end{aligned} \tag{2.9}$$

We put (2.9) into (2.8), and get

$$d(Tx_n, Tx_{n+1}) \le [d(Tx_{n-2}, Tx_{n-1})]^{2k}. \tag{2.10}$$

As evident from Remark 1, the case where $(Tx_n, Tx_{n+1}) \in Q \times Q$ is not possible.

Taking into consideration (2.1), (2.3), (2.7) and (2.10), for all possible cases where $n \ge 2$, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\\ &\le [\max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-1}, Tx_n)\}]^{2k}, \\ &\text{where } k = \frac{\lambda}{1-\lambda}. \end{aligned} \tag{2.11}$$

Because the logarithm function is an increasing function, (2.11) leads to

$$\begin{aligned} &\log\big(d(Tx_n, Tx_{n+1})\big) \\ &\le 2k \max\{\log\big(d(Tx_{n-2}, Tx_{n-1})\big), \log\big(d(Tx_{n-1}, Tx_n)\big)\}. \end{aligned} \tag{2.12}$$

From the assumption, we have $\lambda \in (0, 1/3)$. Because $k = \dfrac{\lambda}{1-\lambda}$, this means $2k \in (0,1)$.

We apply Lemma 1.19 on (2.12), with $w_n = \log\big(d(Tx_n, Tx_{n+1})\big)$ and get

$$\log\big(d(Tx_n, Tx_{n+1})\big) \le (2k)^{n/2}\delta, \tag{2.13}$$

where $\delta = (2k)^{-1/2} \max\{\log\big(d(Tx_0, Tx_1)\big), \log\big(d(Tx_1, Tx_2)\big)\}$.

The exponential function is an increasing function. If we apply it to both sides of (2.13), we get

$$d(Tx_n, Tx_{n+1}) \le \exp\big((2k)^{n/2}\delta\big). \tag{2.14}$$

Let $m, n \in \mathbb{N}$ with $m < n$. Using (m3) of Definition 1.1 inductively, we get

$$\begin{aligned} d(Tx_m, Tx_n) &\le \prod_{i=m}^{n-1} d(Tx_i, Tx_{i+1}) \\ &\le \prod_{i=m}^{+\infty} d(Tx_i, Tx_{i+1}) \\ &\le \prod_{i=m}^{+\infty} \exp\big((2k)^{i/2}\delta\big), \text{ by (2.14)} \\ &= \exp\left(\delta \sum_{i=m}^{+\infty} (2k)^{i/2}\right) \\ &= \exp\left(\delta (2k)^{m/2} \frac{1}{1-(2k)^{1/2}}\right), \text{ sum of G.P.} \end{aligned}$$

We take the limits $m, n \to +\infty$. In doing so, we use the continuity of the exponential function. We also recall that $2k \in (0,1)$. We get

$$\lim_{m,n \to +\infty} d(Tx_m, Tx_n) = 1.$$

By Lemma 1.9, this shows that the sequence $\{Tx_n\} \in C$ is a multiplicative Cauchy sequence. By Theorem 1.13, because $TC$ is a multiplicative closed, it is also complete. This means there is $z \in TC$ such that $Tx_n \to_* z$ as $n \to +\infty$.

Consider the subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ for which $Tx_{n_k} \in P$ for all $k \in \mathbb{N}$. For $n_k \ge 1$, we have

$$\begin{aligned} &Tx_{n_k} = Sx_{n_k - 1} \\ \Rightarrow &\lim_{k \to +\infty} Tx_{n_k} = \lim_{k \to +\infty} Sx_{n_k - 1} = \lim_{k \to +\infty} Sx_{n_k} = z. \end{aligned} \tag{2.15}$$

Because $z \in TC$, there is $u \in C$ such that $Tu = z$.

Using (i) in the assumption, let us consider the following:

$$\begin{aligned} d(Su, Sx_{n_k}) &\le [\max\{d(Tu, Tx_{n_k}), d(Tu, Su), \\ &\qquad d(Tx_{n_k}, Sx_{n_k}), d(Tu, Sx_{n_k}), d(Tx_{n_k}, Su)\}]^\lambda \\ &= [\max\{d(z, Tx_{n_k}), d(z, Su), d(Tx_{n_k}, Sx_{n_k}), \\ &\qquad d(z, Sx_{n_k}), d(Tx_{n_k}, Su)\}]^\lambda. \end{aligned}$$

Taking limits $k \to +\infty$, we get

$$\begin{aligned} d(Su,z) &\leq [\max\{d(z,z),d(z,Su),d(z,z),d(z,z),\\ &\qquad d(z,Su)\}]^{\lambda}\\ &= [d(Su,z)]^{\lambda},\\ &\qquad \text{by (m1) and (m2) of Definition 1.1}\\ \Rightarrow d(Su,z) &= 1, \text{ because } \lambda \in (0,1/3)\\ \Rightarrow Su &= z, \text{ by (m1).} \end{aligned}$$

Thus $Su = Tu = z$, making $u$ a coincidence point of mappings $T$ and $S$.

As $S$ and $T$ are coincidentally commuting, we have

$$\begin{aligned} &STu = TSu\\ &\Rightarrow Sz = Tz. \end{aligned} \tag{2.16}$$

We use (i) in the assumption again and get

$$\begin{aligned} d(Su,Sz) &= d(z,Sz)\\ &\leq [\max\{d(Tu,Tz),d(Tu,Su),d(Tz,Sz),\\ &\qquad d(Tu,Sz),d(Tz,Su)\}]^{\lambda}\\ &= [\max\{d(z,Sz),d(z,z),d(Sz,Sz),d(z,Sz),\\ &\qquad d(Sz,z)\}]^{\lambda}\\ \Rightarrow d(z,Sz) &\leq [d(z,Sz)]^{\lambda}\\ \Rightarrow d(z,Sz) &= 1, \text{ because } \lambda \in (0,1/3)\\ &\Rightarrow Sz = z = Tz, \end{aligned}$$

making $z$ a common fixed point of $S$ and $T$.

We show that $z$ is unique. Suppose $z'$ is also a fixed point of $T$. From the assumption, we have

$$\begin{aligned} d(z,z') &= d(Sz,Sz')\\ &\leq [\max\{d(Tz,Tz'),d(Tz,Sz),d(Tz',Sz'),\\ &\qquad d(Tz,Sz'),d(Tz',Sz)\}]^{\lambda}\\ &= [\max\{d(z,z'),d(z,z),d(z',z'),d(z,z'),\\ &\qquad d(z',z)\}]^{\lambda}\\ &= [d(z,z')]^{\lambda}\\ \Rightarrow d(z,z') &= 1, \text{ because } \lambda \in (0,1/3)\\ &\Rightarrow z = z' \text{ by (m1).} \end{aligned}$$

Hence the fixed point $z$ is unique. □

We now show an example on the use of this theorem.

## 3. Example

Consider the multiplicative space metric space $(\mathbb{R}_+, d)$, with $d = d_a$ defined as in Example 1.3 with $a = 2$. The space $(X,d)$ is complete and multiplicative metrically convex.

Let $C = [0,2]$. We define the mappings $S,T : C \to \mathbb{R}_+$ as follows

$$Sx = \begin{cases} 3x, & x \in [0,1]\\ 1, & x \in (1,2] \end{cases}, \qquad Tx = \begin{cases} 10x, & x \in [0,1]\\ 10/3, & x \in (1,2] \end{cases}.$$

From the given information, $TC = [0,10]$ is closed. We have $\partial C = \{0,2\} \subset TC$. We also have $Tx \in \partial C$ implies $x \in \{0,0.2\}$, which means $Sx \in \{0,0.6\} \subset C$. Furthermore, both $S$ and $T$ are discontinuous at 1.

We note that for some values $c \in C$, (such as $c = 1$), we have $Sc, Tc \notin C$, showing that $S$ and $T$ are non-self mappings.

Without loss of generality let $x,y \in C, x \geq y$.

Suppose $x,y \in [0,1]$. Then

$$\begin{aligned} d(Sx,Sy) &= d(3x,3y)\\ &= 2^{3|x-y|}\\ &= 2^{0.3(10|x-y|)}\\ &= \left[2^{|10x-10y|}\right]^{0.3}\\ &= [d(Tx,Ty)]^{0.3}\\ &\leq \left[\max\{d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),\right.\\ &\qquad \left. d(Tx,Sy),d(Ty,Sx)\}\right]^{\lambda}, \end{aligned}$$

where $\lambda = 0.3 \in \left(0,\frac{1}{3}\right)$.

Suppose $x \in (0,2], y \in [0,1]$. Then

$$\begin{aligned} d(Sx,Sy) &= d(1,3y)\\ &= 2^{|1-3y|}\\ &= 2^{(0.3|10/3-10y|)}\\ &= [2^{|10/3-10y|}]^{0.3}\\ &= [d(Tx,Ty)]^{0.3}\\ &\leq \left[\max\{d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),\right.\\ &\qquad \left. d(Tx,Sy),d(Ty,Sx)\}\right]^{\lambda}, \end{aligned}$$

where $\lambda = 0.3 \in \left(0,\frac{1}{3}\right)$.

Finally, suppose $x,y \in (0,2]$. Then

$$\begin{aligned} d(Sx,Sy) &= d(1,1)\\ &= 2^{|1-1|}\\ &= 2^{|10/3-10/3|}\\ &= [d(Tx,Ty)]^{0.3}\\ &\leq \left[\max\{d(Tx,Ty),d(Tx,Sx),d(Ty,Sy),\right.\\ &\qquad \left. d(Tx,Sy),d(Ty,Sx)\}\right]^{\lambda}, \end{aligned}$$

where $\lambda = 0.3 \in \left(0,\frac{1}{3}\right)$.

The conditions for the assumption have been met, and $z = 0$ is the unique fixed point of $T$ because $S(0) = T(0) = 0$.



## 4. Conclusion

In this paper we have formulated a fixed point theorem for pairs of non-self mappings in multiplicative metric spaces by modifying a study by Khan and Imdad [8], which was developed for metric spaces.

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