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EFFECTS OF BUOYANCY ON THE LOWER BRANCH MODES ON A BLASIUS BOUNDARY LAYER

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Abstract. The effect of thermal buoyancy on the stability properties of lower branch Tollmein–Schlichting waves are investigated. At moderate values of thermal buoyancy the standard triple deck structure, which describes the evolution of such short wavelength instabilities in a buoyant boundary layer, is unaltered. The leading order eigenrelation is now a function of thermal buoyancy and from it we can derive the new dominant length- and time–scales for the instability in the case when the boundary layer is strongly buoyant. These new scales demonstrate that, in the case of strong wall cooling the lower branch structure is identical to the upper branch structure, thus suggesting that the curve of neutral stability may become closed at some large value of a Reynolds number. In the alternate limit of strong wall heating the evolution of a fixed frequency disturbance is governed by the linearized interactive boundary-layer equations; in this case wave–like disturbances cannot be described by any form of the quasi–parallel approximation theory.

1. **Introduction.** Theoretical studies on the stability of flat plate thermal boundary layers to a vortex mode of instability include investigations by Wu and Cheng [15], Moutsoglou, Chen and Cheng [9] and Chen and Cheng [2], who all used the quasi-parallel stability theory. Later Hall and Morris [4] gave a self-consistent account of non-parallel effects for finite values of a buoyancy related parameter, the Grashof number. The flat plate thermal boundary layer is unstable to the vortex mode of instability. Hall and Morris [4] included in their study on convective boundary layers, the effects of free-stream disturbances, wall roughness and non-uniform wall heating on the growth of streamwise vortices. Hall [3] extended the work of Hall and Morris [4] to include nonlinear aspects of vortex instability in a heated boundary layer. Other possible forms of instability in a heated boundary layer include viscous Tollmien-Schlichting waves and inviscid Rayleigh waves.

Investigations on the effects of surface cooling/heating on the stability of compressible boundary layers include the theoretical work by Lees [5], Mack ([6],[7],[8]); amongst others. There is general consensus concerning the destabilizing effect of heating to compressible boundary layers. The earlier studies by Lees [5], Mack ([6], [7]) show that cooling has a stabilizing effect on compressible boundary layers. Later Mack [8] showed that at lower Mach numbers, cooling stabilizes the first mode of instability and destabilizes the inviscid second (and higher) modes. However, Seddougui et al [11] have shown that cooling destabilizes both the first and second modes, with the growth rate of the inviscid mode comparable to, or even

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exceeding, that of the usually more unstable viscous Tollmien-Schilichting modes. There is therefore a discrepancy in the theoretical results of Mack and Seddougui which could perhaps be traced back to the approaches used in getting their results. Mack's work is based on a parallel flow approximation while Seddougui et al used an approach based on triple-deck matched asymptotic which is able to account for the effect of boundary layer growth.

The effect of buoyancy on the upper branch TS modes has been considered by Mureithi *et al* [10] who demonstrated the destabilizing (stabilizing) effect of the strong heating(cooling) on the boundary layer. The work presented here considers the effects of thermal buoyancy on the stability of lower branch TS waves in incompressible boundary layers, and is based upon the triple deck theory. Smith [12] considered the nonlinear stability characteristics of the lower branch of the neutral curve with a triple deck flow structure, for a Blasius type of flow. In this study we focus on the effects of thermal buoyancy on stability properties of the lower branch modes. The study takes care of the non-parallel flow effects through the use of the triple deck theory, at small values of the buoyancy forces. As the buoyancy forces are increased asymptotically, the study looks at the modifications to the triple deck structure.

In section 2 the governing equations for the flow are formulated. In section 3 the disturbance equations governing the flow are given and the leading order eigenrelation derived. For small buoyancy forces we demonstrate that the triple deck flow structure is unaltered but the leading eigenrelation is modified. Positive thermal buoyancy has a destabilizing effect on the flow while negative thermal buoyancy (cooling) is stabilizing. In section 4 we investigate the structural and stability changes that arise in the large buoyancy limit and demonstrate, in the limit of strong cooling, a close relationship between the upper and the lower branch buoyancy modified TS waves. Finally some conclusions are drawn in section 4.

2. **The governing equations.** Consider a two-dimensional viscous incompressible flow over a heated/cooled horizontal wall. The flow has density ρ, kinematic viscosity ν, free-stream velocity U_∞ and free-stream temperature T_∞. Let T_w denote a wall temperature. The non-dimensional Navier–Stokes equations governing the flow, under the Boussinesq approximations, are

$$\nabla.\mathbf{q} = 0, \tag{1}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{q} + \mathbf{F}\,T \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{q}.\nabla)T = \frac{1}{PrRe}\nabla^2 T \tag{3}$$

where $\mathbf{q} = (u, v)$ and $\mathbf{F} = (0, G)$. The equations have been non-dimensionalized using the free–stream velocity U_∞, the characteristic length L and the temperature difference ΔT. The parameters of the flow become the Reynolds number, Re, the Prandtl number, Pr and the buoyancy related parameter G, defined respectively as

$$Re = \frac{U_\infty L}{\nu}, \quad Pr = \frac{\mu C_p}{\kappa}, \quad G = GrRe^{-2}$$

where κ is the coefficient of thermal conductivity, C_p is the coefficient of specific heat of the fluid at constant pressure and Gr is the Grashof number defined as $Gr = g\tilde{\beta}L^3\Delta T\nu^{-2}$, with $\tilde{\beta}$ being the coefficient of volume expansion and g is the acceleration due to gravity.

2.1. **The boundary-layer equations.** In the limit $Re \to \infty$ the flow develops a boundary layer of thickness $O(Re^{-1/2})$ attached to the wall. We define a small parameter $\varepsilon = Re^{-1/8}$.

The equations governing the steady state boundary layer flow over a heated/cooled horizontal wall are obtained by defining the boundary layer scalings

$$y = \varepsilon^4 Y, \quad (u, v, p, T) = [\bar{u}, \varepsilon^4 \bar{v}, \bar{p}, \bar{T}](x, y) + \cdots$$

We expand G as $G = \varepsilon^{-n} G_0$, where $G_0 \sim O(1)$. Substituting these expansions into the Navier–Stokes equations and taking the formal limit $Re \to \infty$ (so that $\varepsilon \ll 1$) gives the boundary-layer equations

$$\begin{aligned} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial Y} &= 0, \\ \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial Y} &= -\frac{\partial \bar{p}}{\partial x} + \frac{\partial^2 \bar{u}}{\partial Y^2}, \\ \frac{\partial \bar{p}}{\partial Y} &= \varepsilon^{4-n} G_0 \bar{T} \\ \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial Y} &= \frac{1}{Pr}\frac{\partial^2 \bar{T}}{\partial Y^2} \end{aligned} \tag{4}$$

with boundary conditions

$$\left.\begin{array}{ll} \bar{u}(x,0) = \bar{v}(x,0) = 0, & \bar{T}(x,0) = T_w(x) \\ \bar{u}(x, Y \to \infty) \to 1, & \bar{T}(x, Y \to \infty) \to 0 \end{array}\right\} \tag{5}$$

It is only when $n = 4$ that the system (4) becomes fully coupled through the $O(1)$ temperature induced pressure appearing in (4).

We initially examine the case when $G \sim O(Re^{3/8})$, that is, $G = \varepsilon^{-3} G_0$. The velocity and temperature fields decouple and hence this case when $n = 3$ will be termed as the moderate level of buoyancy. Then the decoupled boundary-layer equations (4) admit a family of self-similar solutions of the form

$$\bar{u}(x,y) = f'(\eta), \quad \bar{v}(x,y) = -\frac{1}{2\sqrt{x}}\left(f - \eta f'\right), \quad \bar{T}(x,y) = \theta(\eta)$$

where $\eta = yx^{-1/2}$ is a boundary layer similarity variable and $f'(\eta)$ and $\theta(\eta)$ are the stream-wise velocity and temperature similarity variables. The similarity form for the boundary-layer equations and the boundary conditions is

$$2f''' + ff'' = 0, \quad 2\theta'' + Prf\theta' = 0, \tag{6}$$

$$f(0) = f'(0) = \theta(0) - 1 = 0, \quad f'(\infty) = 1, \quad \theta(\infty) = 0. \tag{7}$$

Equation (6(a)) with boundary conditions (7(a)) is the Blasius equation.

The flow has the properties

$$\left.\begin{array}{l} \bar{u} = \lambda Y + O(Y^4) \quad \bar{T} = 1 + \mu Y + O(Y^2) \quad \text{as} \quad Y \to 0, \\ \bar{u} \to 1, \quad \bar{T} \to 0 \quad \text{as} \quad Y \to \infty. \end{array}\right\} \tag{8}$$

where $\lambda = 0.332x^{-1/2}$ is the local skin friction coefficient. Here μ is the local temperature gradient at the wall and is dependent on the Pr number. For the case when $Pr = 0.7$, $\mu = -0.293x^{-1/2}$.

Our concern is with the stability characteristics of the basic flow, governed by (4), as the thermal buoyancy is increased asymptotically from $G \sim O(Re^{3/8})$ to $G \sim O(Re^{1/2})$. In the absence of buoyancy (that is $G = 0$) the energy equation can

effectively be ignored, The flow then is governed by the classical Blasius boundary-layer equations. In this case the problem reduces to that of Smith [12] for the nonlinear stability of the Blasius boundary layer. The lower branch asymptotic structure is the triple deck structure.

3. **The Triple Deck Structure.** For the moderate buoyancy levels when $G = \varepsilon^{-3}G_0$, $G_0 = O(1)$, with Blasius type basic flow. However, it is at this order of G that buoyancy first have an appreciable effect on the standard triple deck results of Smith ([13], [12]), through a modification to the leading order eigenrelation. We will therefore first look at the case $G = O(Re^{3/8})$ where the stability structure is basically the triple deck structure described in details in Smith [12]; later, attention will be given to the strong thermal stratification limit when $G = O(\varepsilon^{-4})$. In the following subsection we look at the expansions and the solutions in the various decks.

At moderate levels of thermal buoyancy, the temporal and spatial scalings are the usual triple deck scalings, given respectively by $t = \varepsilon^2\tau$ and $x = \varepsilon^3 X$, with X and τ being $O(1)$ quantities. The triple deck structure is presented in fig 1.

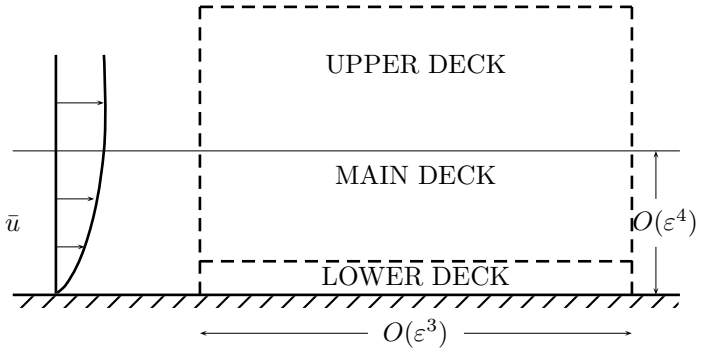


FIGURE 1. The triple-deck lower branch asymptotic structure

We consider disturbances to the basic flow which are proportional to $E = \exp(i\alpha x - i\beta\tau)$, where the wavenumber α and the wave frequency β have the expansions

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \cdots, \quad \beta = \beta_0 + \varepsilon\beta_1 + \cdots$$

Initially when $G = \varepsilon^{-3}G_0$, (with $G_0 \sim O(1)$), the temporal and the spatial scalings are the usual triple deck scalings given respectively by

$$t = \varepsilon^2\tau, \quad x = \varepsilon^3 X, \quad \text{where} \quad X = O(1) \ \text{ and } \ \tau = O(1).$$

We also asymptotically expand the following:

$$x = x_0 + \varepsilon^2 x_1 + \cdots, \quad (\lambda, \mu) = (\lambda_0, \mu_0) + \varepsilon(\lambda_1, \mu_1) + \cdots$$

where $\lambda_0 = 0.332\, x_0^{-1/2}$ and $\mu_0 = -0.293\, x_0^{-1/2}$

3.1. **Main deck:** $y = O(Re^{-1/2})$**.** This layer encompasses most of the boundary layer and is of thickness $O(Re^{-1/2})$. The basic flow is the steady state boundary-layer equations (4) described above. Let us define

$$y = \varepsilon^4 Y, \quad (u, v, p, T) = (\bar{u}, \varepsilon^4 \bar{v}, \bar{p}, \bar{T}) + \varepsilon[u_0, \varepsilon v_0, \varepsilon p_0, \theta_0]E + \cdots, \tag{9}$$

where $Y = O(1)$. Substituting the main deck expansions into the linearized disturbance equations and equating coefficients of ε to zero, and solving the leading order equations gives

$$u_0 = A_0 \bar{u}_Y, \quad v_0 = -i\alpha_0 A_0 \bar{u}, \tag{10}$$

$$\tag{11}$$

$$\theta_0 = A_0 \bar{T}_Y, \quad p_0 = \tilde{P}_0 + G_0 A_0 (\bar{T} - 1), \tag{12}$$

Here $\tilde{P}_0$ is the wall pressure and $A_0(x)$ is an unknown displacement function. In the limit $Y \to 0$ the main deck solutions take the form

$$\begin{aligned} u &= \lambda_0(Y + \varepsilon A_0) + O(\varepsilon^2), \quad T = 1 + \mu_0(Y + \varepsilon A_0) + O(\varepsilon^2), \\ p &= \varepsilon^2[\tilde{P}_0 + G_0 A_0 \mu_0 Y] + O(\varepsilon^3). \end{aligned} \tag{13}$$

while in the limit $Y \to \infty$ we have the free-stream condition

$$u \to 1, \quad T \to 0, \quad p \to \varepsilon^2(\tilde{P}_0 - G_0 A_0). \tag{14}$$

3.2. **Lower deck** $y = O(Re^{-5/8})$**.** The expansions in this deck take the form

$$y = \varepsilon^5 Z, \quad (u, v, p, T) = (\varepsilon\lambda_0 Z, 0, 0, 1 + \varepsilon\mu_0 Z) + \varepsilon(U_0, \varepsilon^2 V_0, \Theta_0, \varepsilon P_0)E + \cdots$$

where $Z = O(1)$. This deck contains a viscous wall layer in which the main deck solutions adjust to the wall no–slip boundary condition.

The equations governing the flow in this deck take the form:

$$\left.\begin{aligned} &i\alpha_0 U_0 + \frac{\partial V_0}{\partial Z} = 0 \\ &\left(\frac{\partial^2}{\partial Z^2} - i\alpha_0\lambda_0\left(Z - \frac{\beta_0}{\alpha_0\lambda_0}\right)\right) U_0 - \lambda_0 V_0 - i\alpha_0 P_0 = 0 \\ &\left(\frac{1}{\mathrm{Pr}}\frac{\partial^2}{\partial Z^2} - i\alpha_0\lambda_0\left(Z - \frac{\beta_0}{\alpha_0\lambda_0}\right)\right)\Theta_0 - \mu_0 V_0 = 0 \end{aligned}\right\} \tag{15}$$

and $dP_0/dZ = 0$. The boundary conditions to be satisfied become

$$U_0 = V_0 = \Theta_0 = 0 \quad \text{on} \quad Z = 0, \tag{16}$$

and matching conditions with the main deck solutions.

Making use of the transformation $Z = a\xi + b$ where $a = (i\alpha_0\lambda_0)^{-1/3}$ and $b = \beta_0/(\lambda_0\alpha_0)$, we have

$$\left(\frac{\partial^2}{\partial\xi^2} - \xi\right)\frac{\partial U_0}{\partial\xi} = 0. \tag{17}$$

which is the Airy equation for $U_{0\xi}$.The solution to the Airy equation is

$$U_0 = C_0 \int_{\xi_0}^{\xi} \mathrm{Ai}(s)ds, \tag{18}$$

where the constant C_0 is an unknown complex-valued amplitude.

3.3. **Upper deck: $y = O(Re^{-3/8})$.** The expansions in this deck take the form

$$y = \varepsilon^3 \hat{y}, \quad (u, v, p) = (1, 0, 0) + \varepsilon^2[(u^{(0)}, v^{(0)}, p^{(0)}) + \cdots]E,$$

with $\hat{y} = O(1)$. The solutions in this deck are

$$p^{(0)} = B_0 e^{-\alpha_0 \hat{y}}, \quad v^{(0)} = -iB_0 e^{-\alpha_0 \hat{y}}$$

The equations have been solved subject to the condition of boundedness as $\hat{y} \to \infty$ together with matching with the main deck solutions in the limit $\hat{y} \to 0$.

4. **Leading order eigenrelation.** Solutions are now matched asymptotically in their respective overlap regions. Matching the main deck solutions as $Y \to \infty$ and upper deck solutions as $\hat{y} \to 0$ gives

$$B_0 = \alpha_0 A_0, \quad \tilde{P}_0 = A_0(G_0 + \alpha_0).$$

Matching the solutions in the lower ($Z \to \infty$) and the main deck ($Y \to 0$) gives

$$A_0 \lambda_0 = C_0 \int_{\xi_0}^{\infty} \mathrm{Ai}(s) ds, \tag{19}$$

$$C_0 \mathrm{Ai}'(\xi_0) = (i\alpha_0)^{2/3} \lambda_0^{-2/3} \tilde{P}_0, \tag{20}$$

Hence the leading order eigenrelation takes the form

$$\lambda_0^{5/3} \mathrm{Ai}'(\xi_0) = (i\alpha_0)^{1/3}(\alpha_0 + G_0)\kappa, \quad \xi_0 = -i^{1/3}\beta_0(\alpha_0\lambda_0)^{2/3} \tag{21}$$

where $\kappa = \int_{\xi_0}^{\infty} \mathrm{Ai}(s) ds$.

For a neutrally stable leading order perturbation we require α_0 and β_0 to be real and hence (see [12], [1])

$$\xi_0 = -2.298 i^{1/3}, \qquad \xi_0 \int_{\xi_0}^{\infty} \mathrm{Ai}(s) ds = -2.296 \mathrm{Ai}'(\xi_0).$$

The lower branch neutral frequency and wavenumber then satisfy

$$\beta_0 = 2.298(\lambda_0 \alpha_0)^{2/3}, \qquad G_0 = 1.001 \lambda_0^{5/3} \alpha_0^{-1/3} - \alpha_0, \tag{22}$$

The neutral wavenumber α_0 and the neutral wave frequency β_0 are shown in figures 2 as functions of G_0. These results show that as $G_0 \to \infty$ the neutral wavenumber and frequency tend to zero; while as $G_0 \to -\infty$, $(\alpha_0, \beta_0) \sim G_0$. Note also that as $G_0 \to 0$ the eigenrelation (22) reduces to that of Smith ([12], [13]).

Perhaps a more effective way of interpreting (22) (and the subsequent limits $|G_0| \to \infty$) is to consider the variation in the *neutral position* of a fixed frequency disturbance. We will restrict our attention to the Blasius boundary layer for which $\lambda = 0.332 x^{-1/2}$ (with $\eta = Y x^{-1/2}$ the usual Blasius similarity variable). For fixed frequency β_0, relation (22b) becomes

$$G_0 = \gamma_0 x_0^{-1} - \gamma_1 x_0^{1/2}, \tag{23}$$

where $\gamma_0 = 0.1672 \beta_0^{-1/2}$ and $\gamma_1 = 0.865 \beta_0^{3/2}$. The expression (23) then serves to determine the neutral position x_0 as a function of β_0 and G_0. A plot of the neutral position x_0 versus G_0 (for fixed frequency $\beta_0 = 1$) is given in figure 3 below.

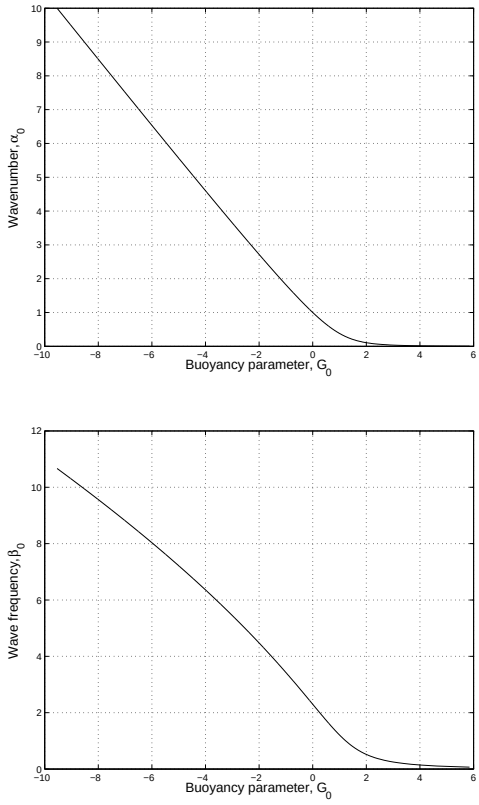


FIGURE 2. The figures (a) and (b) show the neutral wavenumber α_0 and wave frequency β_0 as functions of the buoyancy parameter G_0.

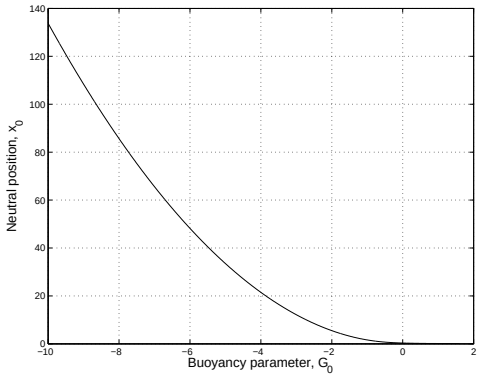


FIGURE 3. Variation of neutral position x_0 with G_0, obtained from solving the equation (23) for fixed frequency $\beta_0 = 1.0$.

4.1. **Large G_0 limit: fixed x_0.** Some analysis concerning the limiting behaviour of the neutral wavenumber, α_0, frequency, β_0, at a fixed neutral position x_0, as the buoyancy parameter G_0 is further increased, is carried out. From the eigenrelation

(22) we have in the limit $G_0 \to \infty$

$$\alpha_0 = \Delta^3\lambda^5 G_0^{-3} + O(G_0^{-7}), \quad \beta_0 = 2.298\lambda^{2/3}\Delta^2\left[G_0^{-2} + O(G_0^{-6})\right], \tag{24}$$

whereas in the limit $G_0 \to -\infty$ we have

$$\left.\begin{aligned} \alpha_0 &= |G_0| + \Delta|G_0|^{-1/3} + O(|G_0|^{-5/3}), \\ \beta_0 &= 2.298\Delta^2\lambda^4\left(|G_0|^{2/3} + \tfrac{2}{3}\Delta|G_0|^{-2/3} + O(|G_0|^{-2})\right). \end{aligned}\right\} \tag{25}$$

where $\Delta = 1.001$. Thus, in the limit $G_0 \to \infty$ (corresponding to increasing the buoyancy force through, for example, an increase in the wall temperature) the neutral frequency and wavenumber are reduced whereas in the limit $G_0 \to -\infty$ (corresponding to an increase in wall cooling) the neutral wavenumber and frequency increase. The limits in (24) and (25) hold until the next distinguished limit for G_0 is reached. An analysis of the original boundary-layer equations suggests that this new limit will arise when $G_0 = O(\varepsilon^{-1})$, at which point the velocity and temperature fields within the boundary layer become fully coupled.

4.2. **Large G_0 limit: fixed β_0.** The effect of increasing G_0 on the neutral position is readily seen from figure 3. In particular we have the limits

$$x_0 = \gamma_0 G_0^{-1} + O(G_0^{-3/2}), \quad \text{as} \quad G_0 \to \infty,$$

$$x_0 = (0.8646)^{-1}|G_0|^2 + O(|G_0|^{-1/2}), \quad \text{as} \quad G_0 \to -\infty.$$

Thus, as the buoyancy parameter is increased the neutral position moves closer to the leading edge of the plate whereas, as we increase the wall cooling ($G_0 \to -\infty$) the neutral position is moved progressively further from the leading edge. Hence increasing G in the positive direction destabilizes the boundary layer, with the reverse occurring as G increases in the negative direction.

A new stability structure arises in this large G_0 limiting case which will be discussed in § 5.

5. **Structure modifications at large G_0 limits.** We now investigate the effects of large G_0 on the stability of the flow under study.

Our analysis has demonstrated that, when the temperature and velocity fields within the boundary layer become fully coupled, then (in the notation above) $G = O(Re^{1/2})$ or $G_0 \sim \varepsilon^{-1}$, and we find that the wavenumber and the wave frequency are such that

$$\begin{aligned} (\alpha_0, \beta_0) &\sim O(\varepsilon^3, \varepsilon^2) \quad \text{as} \quad G_0 \sim \varepsilon^{-1}, \\ (\alpha_0, \beta_0) &\sim O(\varepsilon^{-1}, \varepsilon^{-2/3}) \quad \text{as} \quad G_0 \sim -\varepsilon^{-1}. \end{aligned}$$

This then implies that at asymptotically large G_0 a new stability structure will emerge in which the temporal and streamwise scalings are:

$$(x, t) \sim O(1), \;\; G_0 \sim \varepsilon^{-1},$$

$$(x, t) \sim O(\varepsilon^4, \varepsilon^{8/3}), \;\; G_0 \sim -\varepsilon^{-1}$$

To consider the implications of these new scales let us first define $G_0 = \varepsilon^{-1}\hat{G}_0$ so that $G = \varepsilon^{-4}\hat{G}_0$ where $\hat{G}_0 \sim O(1)$. When $G = O(Re^{1/2})$ the boundary-layer

equations take the form

$$\begin{aligned} \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial Y} &= 0, \\ \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial Y} &= \pm \hat{G}_0 \frac{\partial}{\partial x}\int_Y^\infty \bar{T} dY + \frac{\partial^2 \bar{u}}{\partial Y^2}, \\ \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial Y} &= \frac{1}{Pr}\frac{\partial^2 \bar{T}}{\partial Y^2}, \end{aligned} \tag{26}$$

together with the boundary conditions

$$\left.\begin{array}{llll} \bar{u} = 0, & \bar{T} = 1, & \text{on} & Y = 0, \\ \bar{u} \to 1, & \bar{T} \to 0 & \text{as} & Y \to \infty \end{array}\right\} \tag{27}$$

Equations (26) together with their boundary conditions (27), do not admit self-similar solution. However, due to their parabolic nature, they can be solved numerically by a downstream marching procedure.

5.1. **Large positive G_0.** The scalings are:

$$(x, y, t) = (x, \varepsilon^4 Y, t), \quad (x, Y, t) = O(1)$$

Both the streamwise length and temporal scales are of order one. The flow structure is now the usual two layered boundary layer.

The perturbation expansions now evolve over the same length and spatial scalings as the basic boundary layer scalings. If a small amplitude perturbation imposed on the basic flow grows and becomes of $O(1)$ then the basic flow and the disturbance will be of comparable size. Then the governing equations become the fully nonlinear interactive boundary layers which pose considerable numerical challenge.

5.2. **Strong wall cooling limit.** In this case we have the scalings

$$(x, y, t) = (\varepsilon^4 X, \varepsilon^4 Y, \varepsilon^{8/3}\tau), \quad (x, Y, t) = O(1)$$

In this case the disturbances evolve over a length–scale of $O(\varepsilon^4)$ (equivalent to the boundary layer thickness) and have frequency of $O(\varepsilon^{-8/3})$.

The perturbation expansions take the form:

$$(u, v, T, p) = (u_0, v_0, \theta_0, p_0)E + \cdots, \quad E = \exp(i\alpha_0 X - i\beta_0 \tau)$$

Substituting these expansions into the governing equations gives the stability equation in the form of the steady Taylor-Goldstein equation. This is precisely the results obtained by Mureithi et al [10] for the case of the upper branch TS waves in a highly cooled boundary layer flow. Thus we conclude that in the limit of strong wall cooling the upper and the lower branch of the curve of neutral stability become closed at some asymptotically large value of the Reynolds number. The details of the analysis of the new structure are identical to that described by Mureithi et al [10] and hence not included here.

6. **Conclusion.** The effects of buoyancy on the lower branch stability characteristics of boundary layer flows has been investigated. For the case when $G = O(Re^{3/8})$, the triple deck stability structure is itself unaltered, and only the disturbance solutions and hence the eigenrelations are modified.

For the strongly unstably stratified case (large positive G) the temporal and streamwise scalings are now of order one. The resulting stability equations are the full interactive boundary-layer equations which pose a considerable computational challenge.

For the strongly stably stratified case (large negative G) a two-layer structure evolves. The stability of a particular boundary layer is governed by the steady Taylor–Goldstein equation coupled to an essentially passive viscous wall layer. The disturbance structure so found is identical Mureithi *et al* [10] for the upper branch TS waves modified by strong wall cooling. This suggests that the neutral curve becomes closed and there will be a critical value of a Reynolds number beyond which the flow will be stable to wave–like disturbances. The question of the stability of such flows, at large but finite Reynolds numbers, will then be best decided by recourse to a direct numerical simulation.

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