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SOME FIXED POINT THEOREMS FOR ITERATED CONTRACTION MAPS

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ABSTRACT

The study of iterated contraction was initiated by Rheinboldt in 1969. The concept of iterated contraction proves to be very useful in the study of certain iterative process and has wide applicability. In this paper a brief introduction of iterated contraction maps is given and some fixed point results are discussed.

1. INTRODUCTION

In 1969, Rheinboldt [4] gave unified convergence theory for a class of iterative processes. He initiated the process of finding solution of the nonlinear operator equation $Fx = 0$ using metric fixed point theory. Rheinboldt developed a general convergence theory for iterative processes of the form $x_{k+1} = Gx_k, k = 0, 1, ...;$ and founded on certain nonlinear estimates for the iteration function G as well as on a so called concept of majorizing sequences, His new approach reduces the study of the iterative process to that of a second order nonlinear difference equation.

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The purpose of this paper is to establish yet another fixed point theorem for iterated contractions maps in different settings. Our results extends several results contained in [1], [2], [6] etc.

2. PRELIMINARIES

If f is a contractive mapping, then under certain conditions the sequence $x_{n+1} = fx_n$ tends to the unique fixed-point of f. Ortega, J. M. and Rheinboldt, W. C. [3] proved this type results. The following definitions are useful to our discussion.

Definition 2.1 If $f : X \to X$ is map such that $d(fx, ffx) \leq k\ d(x, fx)$ for all $x \in X, 0 \leq k < 1$, then f is said to be an iterated contraction map.

In case $d(fx, ffx) < d(x, fx), x \neq fx$, then f is an iterated contractive map. See for further results [3,4].

Definition 2.2 A map $f : X \to X$, satisfying $d(fx, fy) \leq kd(x, y), 0 \leq k < 1$, for all $x, y \in X$, is called a contraction map.

A contraction map is continuous and is an iterated contraction.

For example, if $y = fx$, then $d(fx, ffx) \leq kd(x, fx)$ is satisfied. However, converse is not true.

If $f : [-1/2, 1/2] \to [-1/2, 1/2]$ is given by $fx = x^2$, then f is an iterated contraction but not a contraction map.

If $f : R \to R$, defined by $fx = 0$ for $x \in [0, 1/2)$ and $fx = 1$ for $x \in [1/2, 1]$, then f is not continuous at $x = 1/2$, and f is an iterated contraction.

Remark 2.1 A contraction map has a unique fixed point. However, an iterated contraction map may have more than one fixed point.

For example, the iterated contraction function $fx = x^2$ on $[0, 1]$ has $f0 = 0$ and $f1 = 1$, two fixed points.

Remark 2.2 If $f : R \to R$ is given by $fx = 2x+1$, then f is a continuous map but f is not an iterated contraction. It is easy to see that $d(fx, ffx) \leq kd(x, fx)$ is not satisfied for $0 \leq k < 1$.

Remark 2.3 A discontinuous function need not always be an iterated contraction.

Let $X = [0, 1]$. Define $f : [0, 1] \to [0, 1]$ by $fx = 1/2, x \in [0, 1/2)$, and

$fx = 0, x \in [1/2, 1]$. Then f is discontinuous.
If we take $x = 1/4$, then $d(fx, ffx) \leq kd(x, fx), 0 \leq k < 1$, is not satisfied and hence f is not an iterated contraction.

If $f : [0, 1] \to [0, 1]$ defined by $fx = 1/4, x \in [0, 1/2)$, and $fx = 3/4, x \in [1/2, 1]$,then f is discontinuous at $x = 1/2, f$ has fixed points when $x = 1/4$ and $x = 3/4$.

If $f : R \to R$ defined by $fx = x/3 + 1/3$ for $x \leq 0$, and $fx = x/3$ for $x > 0$, then f is an iterated contraction and f has no fixed point.

If $f : [0, 1] \to [0, 1]$ is defined by $fx = 0$ for $x \in [0, 1/2)$, and $fx = 1/2$ for $x \in [1/2, 1]$, then f is discontinuous at $x = 1/2$ and f is an iterated contraction map and has a fixed point $f(1/2) = 1/2$.

The following is a fixed point theorem for iterated contraction map.

Theorem 2.1 If $f : X \to X$ is a continuous iterated contractive map and the sequence of iterates $\{x_n\}$ defined by $x_{n+1} = fx_n, n = 1, 2, ...$ for $x \in X$, has a subsequence converging to $y \in X$, then $y = fy$, that is,f has a fixed point.

Proof: Let $x_{n+1} = fx_n, n = 1, 2, ...$ Then the sequence $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence of reals. It is bounded below by 0, and therefore has a limit. Since the subsequence converges to y and f is continuous on X, so $f(x_{n_i})$ converges to fy and $ff(x_{n_i})$ converges to ffy.

Thus $d(y, fy) = lim\ d(x_{n_i}, x_{n_{i+1}}) = lim\ d(x_{n_{i+1}}, x_{n_{i+2}}) = d(fy, ffy)$.
If $y \neq fy$, then $d(ffy, fy) < d(fy, y)$, since f is an iterative contractive map. Consequently, $d(fy, y) = d(ffy, fy) < d(fy, y)$, a contradiction and hence $fy = y$.

We give the following example to show that if f is an iterated contraction that is not continuous, then f may not have a fixed point.

Let $f : R \to R$, given by $fx = x/5 + 1/5$ for $x \leq 0$, and $fx = x/5$ for $x > 0$. Then f does not have a fixed point. Here f is discontinuous at $x = 0$. If f is continuous but X is not complete, then f need not have a fixed point.

For example, if $f : (0, 1] \to (0, 1]$ given by $fx = x/2$, then f is an iterated contraction, but has no fixed point.

A continuous function f that is not an iterated contraction may not have a fixed point.

The following example illustrates the fact:
If $f : R \to R$ is a translation map $fx = x + 2$, then f has no fixed point but f is continuous.

Note 2.1 If f is not contraction but some powers of f is contraction, then f has a unique fixed point on a complete metric space.

For example, if $f : R \to R$ is defined by $fx = 1$ for x rational and $fx = 0$ for x irrational, then $fofx = f^2x = 1$ a constant map, is a contraction map, and f has a unique fixed point.

Note 2.2 Continuity of an iterated contraction is sufficient but not necessary.

Let $f : [0, 1] \to [0, 1]$ be defined by $fx = 0$ on $[0, 1/2)$, and $fx = 2/3$, for $x \in [1/2, 1]$. Then f is an iterated contraction and $f0 = 0, f2/3 = 2/3$, and f is not continuous.

As stated in Note 2.1 that if f is not contraction still f may have a unique fixed point when some powers of f is a contraction map. The same is true for iterated contraction map.

Theorem 2.2 Let $f : X \to X$ be an iterated contraction map on a complete metric space X. If for some power of f, say f^r is an iterated contraction, that is, $d(f^r x, f^r f^r x) \leq k\ d(x, f^r x)$ and f^r is continuous at y, where $y = lim(f^r)^n x$, for any arbitrary $x \in X$. Then f has a fixed point.

Proof: Since X is a complete metric space and f is an iterated contraction that is continuous at y, therefore $y = f^r y$, where $y = lim(f^r)^n x$.

It is easy to show that $d(y, fy) \leq k^r\ d(y, fy)$. Since $k^r \leq 1$, therefore $d(y, fy) = 0$ and hence f has a fixed point.

We give the following example to illustrate the theorem.

Example 2.1 If $f : [0, 1] \to [0, 1]$ is given by $fx = 1/4, x \in [0, 1/4], f(x) = 0, x \in (1/4, 1/2)$, and $fx = 1/2, x \in [1/2, 1]$, then f is not continuous at $x = 1/4$. However, if we take iteration, then $f^r(1/4) = 1/4$ on $[0, 1/2)$ and $f^r(1/4) = 1/2$ on $[1/2, 1]$ for $r = 2, 3, ...$ It is clear that

$f(1/4) = 1/4$.

Note 2.3 If f is not an iterated contraction in Theorem 2.2, but f^r is an iterated contraction with $f^r y = y$, then f has a fixed point.

The following example is worth mentioning.

Example 2.2 Let $f : [0,1] \to [0,1]$ defined by $fx = 1/4$ for $x \in [0,1/4]$, $fx = 0$ for $x \in (1/4,1/2)$, $fx = 1/2$ for $x \in [1/2,3/4]$ and $fx = 3/4$ for $x \in (3/4,1]$.

Here f is not iterated contraction but f^r is iterated contraction for $r = 2,3,...f^2(x) = 1/4$ for $x \in [0,1/2)$ and $f^2(x) = 1/2$ for $x \in [1/2,1]$. In this example f has a fixed point at $x = 1/4$.

3. MAIN RESULTS

In this section, we will prove some fixed point theorems for iterated maps under different settings. Our results will corollarize many existing results. The first theorem is as follows:

Theorem 3.1 If $f : X \to X$ is an iterated contraction map, and X is a complete metric space, then the sequence of iterates x_n converges to $y \in X$. In case, f is continuous at y, then $y = fy$, that is,f has a fixed point.

Proof: Let $x_{n+1} = fx_n, n = 1,2...; x_1 \in X$. It is easy to show that $\{x_n\}$ is a Cauchy sequence, since f is an iterated contraction. The Cauchy sequence $\{x_n\}$ converges to $y \in X$, since X is a complete metric space. Moreover, if f is continuous at y, then $x_{n+1} = fx_n$ converges to fy. It follows that $y = fy$.

Note 3.1 A continuous iterated contraction map on a complete metric space has a unique fixed point. If an iterated contraction map is not continuous, even then it may have a fixed point.

For example, if $f : [0,1] \to [0,1]$ is given by $fx = 0$ for $x \in [0,1/2)$ and $fx = 2/3$ for $x \in [1/2,1]$. In this example, $X = [0,1]$ is complete but f is not continuous at $x = 1/2$. However, f is continuous at $x = 0, f0 = 0$ and f is continuous at $x = 2/3, f2/3 = 2/3$.

Theorem 3.2 If $f : C \to C$ is an iterated contraction and is continuous, where C is closed subset of a metric space X, then f has a fixed point provided that $f(C)$ is compact.

Proof: We show that the sequence $\{x_n\}$ has a convergent subsequence. Using iterated contraction and continuity of f we get a fixed point.

Definition 3.1 Let X be a metric space and $f : X \to X$. Then f is said to be an iterated nonexpansive map if

$$d(fx, ffx) \leq d(x, fx)$$

for all $x \in X$.

The following is a fixed point theorem for the iterated nonexpansive map.

Theorem 3.3 Let X be a metric space and $f : X \to X$ an iterated nonexpensive map satisfying the following :
(i) if $x \neq fx$, then $d(ffx, fx) < d(fx, x)$,
(ii) if for some $x \in X$, the sequence of iterates $x_{n+1} = fx_n$ has a convergent subsequence converging to y say and f is continuous at y. Then f has a fixed point.

Proof: It is easy to show that the sequence $\{d(x_{n+1}, x_n)\}$ is a nonincreasing sequence of positive reals bounded below by 0. The sequence has a limit. Hence, $d(fy, y) = lim\ d(x_{n_i}, x_{n_{i+1}}) = lim\ d(x_{n_{i+1}}, x_{n_{i+2}}) = d(ffy, fy)$.
This is a contradiction to (i). Therefore f has a fixed point, that is, $fy = y$.

If C is a compact subset of a metric space X and $f : C \to C$ an iterated nonexpansive map, satisfying condition (i) of the above theorem, then f has a fixed point.

Note 3.2 If C is compact, then condition (ii) of Theorem 3.3 is satisfied, and hence the result.

If C is a closed subset of a metric space X and $f : C \to C$ an iterated contraction. If the sequence $\{x_{n_n}\}$ converges to y, where f is continuous at y, then $fy = y$, that is, f has a fixed point.

The following theorem deals with two metrics on X.

Theorem 3.4 Let $f : X \to X$ satisfy the following:
(i) X is complete with metric d and $d(x, fx) \leq \delta(x, fx)$ for all $x, fx \in X$,
(ii)f is iterated contraction with respect to δ,

Then for $x \in X$, the sequence of iterates $x_n = f^n x$ converges to $y \in X$. If f is continuous at y, then f has a fixed point, say $fy = y$.

Proof: It is easy to show that $\{x_n\}$ is a Cauchy sequence with respect to δ. Since $d(x, fx) \leq \delta(x, fx)$, therefore $\{x_n\}$ is a Cauchy sequence with respect to d. The sequence $\{x_n\}$ converges to y in (X, d) since it is complete. The function f is continuous at y, therefore $fy = f\ lim\ x_n = lim\ fx_n = lim\ x_{n+1} = y$. Hence the result.

Kannan [2] considered the following map.

Let $f : X \to X$ satisfy $d(fx, fy) \leq k[d(x, fx) + d(y, fy)]$ for all $x, y \in X$ and $0 \leq k < 1/2$. Then f is said to be a Kannan map.

If $y = fx$, then we get $d(fx, ffx) \leq k[d(x, fx) + d(fx, ffx)]$. This gives $d(fx, ffx) \leq \frac{k}{(1-k)} d(x, fx)$, where $0 \leq \frac{k}{(1-k)} < 1$, that is, f is an iterated contraction. A Kannan map is an iterated contraction map.

The following result due to Kannan [2] is valid for iterated contraction map.

Theorem 3.5 Let $f : X \to X$ satisfy $d(fx, fy) \leq k[d(x, fx) + d(y, fy)]$ for all $x, y \in X, 0 \leq k < 1/2, f$ continuous on X and let the sequence of iterates $\{x_n\}$ have a subsequence $\{x_{n_i}\}$ converging to y. Then f has a fixed point.

The following result deals with two mappings.

Theorem 3.6 Let $f : X \to X$ and $g : X \to X$, where X is a complete metric space, satisfy $d(fx, gfx) \leq kd(x, fx)$ and $d(gx, fgx) \leq kd(x, gx)$ for some $k, 0 \leq k < 1$ and for each $x_1 \in X$. If f and g are continuous on X, then f and g have a common fixed point.

Proof: We consider the sequence of iterates as follows. Let $x_2 = fx_1, x_3 = gx_2, x_4 = fx_2$, and so on. In this case it is easy to show that the sequence $\{x_n\}$ is a Cauchy sequence and converges in X since X is a complete metric space.

Let $lim\ x_n = y$. Then $lim\ x_{2n} = y$ and $lim\ x_{2n-1} = y$. Since f and g are continuous on X, therefore,

$$fy = f\ lim\ x_{2n-1} = lim\ fx_{2n-1} = lim\ x_{2n} = y.$$

$$gy = g\ lim\ x_{2n} = lim\ gx_{2n} = lim\ x_{2n+1} = y.$$

Hence $y = fy = gy$ and y is a common fixed point.

Results of this type for topological vector spaces are given in [5]. We are giving the following:

Theorem 3.7 Let $f : X \to X$ be a continuous iterated contraction map such that:

(i) if $x \neq fx$, then $d(fx, ffx) < d(x, fx)$, and

(ii) the sequence $x_{n+1} = f(x_n)$ has a convergent subsequence converging to y(say).

Then the sequence $\{x_n\}$ converges to a fixed point of f.

Proof: It is easy to see that the sequence $\{d(x_n, x_{n+1})\}$ is a nonincreasing and bounded below by 0. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging to y. Then,

$$d(y, fy) = lim\ d(x_{n_i}, x_{n_{i+1}}) = d(x_{n_{i+1}}, x_{n_{i+2}}) = d(fy, ffy) < d(y, fy).$$

This is a contradiction so $y = fy$. Since $d(x_{n+1}, y) < d(x_n, y)$ for all n, so $\{x_n\}$ converges to $y = fy$.

The result due to Cheney and Goldstein [1] follows as a corollary.

Corollary 3.1 Let f be a map of a metric space X into itself such that

(i) f is a nonexpansive map on X, that is, $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$,

(ii) if $x \neq fx$, then $d(fx, ffx) < d(x, fx)$,

(iii) the sequence $x_{n+1} = f(x_n)$ has a convergent subsequence converging to y(say). Then the sequence $\{x_n\}$ converges to a fixed point of f.

The following is well known for contractive maps.

Note 3.3 If $f : X \to X$ is a contractive map and $f(X)$ compact, then f has a unique fixed point.

It is easy to see that the sequence of iterates $\{x_n\}$ converges to a unique fixed point of f. However, for nonexpansive map, a sequence of iterates need not converge to a fixed point of f. For example, if $fx = -x$, then the sequence of iterates $\{x_n\}$ does not converge to a fixed point of $f(f0 = 0)$.

Note 3.4 If $g = afx + (1 - a)x, 0 < a < 1$, then the fixed point of f is the same as of g.

Let $fy = y$. Then $gy = afy + (1 - a)y$, that is, $gy = y$ since $fy = y$.

Let $gy = y$. Then we show that $y = fy$. Here $gy = afy + (1-a)y = y$. Then $fy = y$, that is, f has a fixed point y [6]. In case the sequence $x_{n+1} = gx_n$ converges to y a fixed point of g, then $y = fy$.

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