COMMON FIXED POINTS FOR FOUR NON-SELF-MAPPINGS

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Abstract
In this paper, we formulate a quasi-contraction type non-self mapping on Takahashi convex metric spaces and common fixed point theorems that applies to two pairs of mappings. The result generalizes the fixed point theorems of some previous authors.

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1. Introduction and Preliminaries

Gajić and Rakočević [1] proved a quasi-contraction common fixed point theorem for non-self mappings on Takahashi convex metric spaces for a pair of mappings. In their work they generalized the theorems by Jungck [2], Das and Naik [3], Ćirić et al. [4], Ćirić [5] and Imdad and Kumar [6]. In this study, we extend the theorem by Gajić and Rakoc□ević [1] to apply for two pairs of mappings in metric spaces.

The following are the preliminaries required in this paper.
Given two non-self mappings $f, g: K \to X$ we say that $x \in K$ is *coincidence point* if $fx = gx$. We term the point $y \in X$ as a *point of coincidence* if $y = fx = gx$ where x is a coincidence point. We also say that f and g are *coincidentally commuting* if $fgx = gfx$ whenever x is a coincidence point.

If K is a subset of X, we denote the boundary of K as δK.
Here, we provide the definition of a Takahashi convex metric space which is useful for future discussion.
Definition 1.1. [7]. *Let X be a metric space and $I = [0, 1]$ be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$; $\lambda \in I$,*

$$d\big(u, W\ (x, y, \lambda\ )\big) \le d(u, x) + d(u, y)$$

for every $u \in X$. The metric space (X, d), together with the convex structure is called the Takahashi convex metric space.

If (X, d) is a Takahashi convex metric space, then for every $x, y \in X$, we term

$$\text{seg}[x, y] := \{W(x, y, \lambda) : \lambda \in [0;\ 1]\}.$$

We will use the following property for a Takahashi convex structure in a metric space (X, d).
Lemma 1.2. [7] Let $x, y \in X$ and $z \in \text{seg}[x. y]$, then for all $u \in X$ we have

$$d(u, z) \le \max\{d(u, x), d(u, y)\}.$$

In Gajić and Rakoc□ević [1], the following theorem was proved:

Theorem 1.3. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* C *be a non-empty closed subset of* X *and* δC *be the boundary of* C. *Let* $g, f: C \to X$ *and suppose* $\delta C \neq \emptyset$. *Let us assume that* f *and* g *satisfy the following conditions:*

(i) *For every* $x, y \in C, d(gx, gy) \leq M_\omega(x, y)$ *where*
$M_\omega(x, y) =$
$\max\{[\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)], \omega_4[d(fx, gy)], \omega_5[d(gx, fy)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$,

(ii) $\delta C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \delta C \Rightarrow gx \in C$ *and*

(v) $f(C)$ *is closed in* X.

Then there exists a coincidence point v *in* C. *Moreover, if* f *and* g *are coincidentally commuting, then* v *remains a unique common fixed point of* f *and* g.

2. Results

This paper seeks to modify Theorem 1.3 to four non-self maps. We seek to prove the following theorem.

Theorem 2.1. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X. *Let* δK *be the boundary of* K *with* $\delta K \neq \emptyset$. *Let mappings* $A, B, S, T : K \to X$. *Assume that* A, B, S *and* T *satisfy the following conditions:*

(i) *For every* $x, y \in K, d(Ax, By) \leq M_\omega(x, y)$, *where*
$M_\omega(x, y)$
$= max\{[\omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(By, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, By)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < \frac{1}{2}r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$,

(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,

(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,

(iv) $A(K) \cap K \subset T(K), B(K) \cap K \subset S(K)$ *and*

(v) $S(K), T(K)$ *are closed in* X.

Then there exists a coincidence point $z \in K$ *for* A, B, S *and* T. *Moreover, if each of the pairs* $\{A, S\}$ *and* $\{B, T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, B, S *and* T.

Proof. Commencing with an arbitrary point $w \in \delta K$, we construct a sequence $\{x_n\}$ of points in K as follows:
From assumption (ii), there is a point $x_0 \in K$ such that $Sx_0 = w$. From (iii), $Ax_0 \in K$. According to (iv), we find $x_1 \in K$ such that $Tx_1 = Ax_0$. We locate Bx_1. We consider two scenarios.

(1) If $Bx_1 \in K$, then, using (iv), we can locate $x_2 \in K$ such that $Bx_1 = Sx_2$. We then find Ax_2. If it happens $Ax_2 \in K$, then, from (iv), we can find $x_3 \in K$ such that $Ax_2 = Tx_3$. If however $Ax_2 \notin K$, because W is continuous in the third variable, there is $\lambda_{22} \in (0,1)$ such that $W\ (Sx_2, Ax_2, \lambda_{22}) \in \text{seg}[Sx_2, Ax_2] \cap \delta K$. As $W(Sx_2, Ax_2, \lambda_{22}) \in \delta K$, by (ii), there is $x_3 \in K$ such that $Tx_3 = W(Sx_2, Ax_2, \lambda_{22}) \in \delta K$.

(2) (2) In the case where $Bx_1 \notin K$, because W is continuous in the third variable, there is $\lambda_{11} \in (0,1)$ such that $W(Tx_1, Bx_1, \lambda_{11}) \in \text{seg}[Tx_1, Bx_1] \cap K$. As $W(Tx_1, Bx_1, \lambda_{11}) \in \delta K$, by (ii), there is $x_2 \in K$ such that $Sx_2 = W(Tx_1, Bx_1, \lambda_{11}) \in \delta K$.

In general, we construct the rest of the sequence by proceeding inductively using the following procedure. If $Ax_{2n} \in K$, then, by (iv), we choose $x_{2n+1} \in K$ such that $Tx_{2n+1} = Ax_{2n}$. Similarly if $Bx_{2n+1} \in K$, then, by (iv), we choose $x_{2n+2} \in K$ such that $Sx_{2n+2} = Bx_{2n+1}$.

If however $Ax_{2n} \notin K$, it means, by (iii), there is $\lambda_{2n,2n} \in (0,1)$ and we can choose $x_{2n+1} \in K$ such that $Tx_{2n+1} = W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in K$.
Similarly if $Bx_{2n+1} \notin K$, it means there is $\lambda_{2n+1,2n+1} \in (0,1)$ and we can choose $x_{2n+2} \in K$ such that $Sx_{2n+2} = W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \delta K$.
Now we first prove that

$$Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_n \tag{2.1}$$

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \delta K$, which by (iii) means $Ax_{2n} \in K$. By (iv), this implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction. Using a similar argument we have

$$Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1} \tag{2.2}$$

We now prove that the sequences $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geq 1$, let

$$D_n = \left(\bigcup_{i=0}^{n-1} \{Ax_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Sx_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Tx_{2i+1}\}\right).$$

Let $\alpha_n = \text{diam}(D_n)$. We want to show that

$$\alpha_n \leq \max\{d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j+1}\}, \; 0 \leq j \leq n-1 \tag{2.3}$$

Let us consider the case where $\alpha_n = 0, n \geq 1$.
If $\alpha_n = 0$, we have $Sx_0 = Ax_0 = Bx_1 = Tx_1$. We shall show that Sx_0 is a common fixed point of A and S. As the mappings A and S are coincidentally commuting at the coincidence point x_0, we have

$$Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0 \tag{2.4}$$

From (i), we have for some $t = 1, 4$ or 5,

$$\begin{aligned}
d(SSx_0, Sx_0) = d(ASx_0, Bx_1) &\leq M_\omega(Sx_0, x_1) \\
&= \max\{\omega_1[d(SSx_0, Tx_1)], \omega_2[d(ASx_0, SSx_0)], \omega_3[d(Bx_1, Tx_1)], \\
&\qquad \omega_4[d(ASx_0, Tx_1)], \omega_5[d(SSx_0, Bx_1)]\} \\
&= \max\{\omega_1[d(SSx_0, Sx_0)], \omega_2[d(SSx_0, SSx_0)], \omega_3[d(Sx_0, Sx_0)], \\
&\qquad \omega_4[d(SSx_0, Sx_0)], \omega_5[d(SSx_0, Sx_0)]\} \\
&= \omega_t[d(SSx_0, Sx_0)] \\
&< \frac{1}{2} d(SSx_0, Sx_0) \text{ for } d(SSx_0, Sx_0) > 0 \\
&\Rightarrow \; d(SSx_0, Sx_0) = 0 \\
&\Rightarrow \; SSx_0 = Sx_0.
\end{aligned}$$

Hence Sx_0 is a fixed point of S. From (2.4), we have $SSx_0 = ASx_{0,}$ which implies $d(ASx_0, Sx_0) = 0$, making Sx_0 a fixed point of A too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of T and B. Hence, $z = Sx_0$ is a common fixed point of all four mappings A, B, S and T.

To show the uniqueness of the fixed point, let z' be also a fixed point of A, B, S and T. Then for some $i = 1, 4$ or 5,

$$
\begin{aligned}
d(z,z') &= d(Az, Bz') \\
&\leq \max\{\omega_1[d(Sz, Tz')], \omega_2[d(Az, Sz)], \omega_3[d(Bz', Tz')], \\
&\qquad \omega_4[d(Az, Tz')], \omega_5[d(Sz, Bz')],\} \\
&= \max\{\omega_i[d(z,z')]\} \\
&< \frac{1}{2} d(z,z') \text{ for } d(z,z') > 0 \\
&\Rightarrow d(z,z') = 0 \\
&\Rightarrow z = z'.
\end{aligned}
$$

Hence when $\alpha_n = 0, z = Sx_0$ is the unique common fixed point of A, B, S and T.

We now consider the cases when $\alpha_n > 0$.

Case 1: Consider the case where $\alpha_n = d(Sx_{2i}, Ax_{2j})$ for some $0 \leq i, j \leq n-1$..

Subcase (1. i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ we have for some $s \in \{1, 2, \ldots, 5\}$

$$
\begin{aligned}
\alpha_n = d(Sx_{2i}, Ax_{2j}) &= d(Ax_{2j}, Bx_{2i-1}) \\
&\leq M_\omega(x_{2j}, x_{2i-1}) \\
&\leq \omega_s(\alpha_n) \\
&< \tfrac{1}{2}\alpha_n,
\end{aligned}
$$

which is a contradiction. Hence $i = 0$.

Subcase (1.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it implies $Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and hence by Lemma 1.2 we have

$$\alpha_n = d(Sx_{2i}, Ax_{2j}) \leq \max\{d(Ax_{2j}, Bx_{2i-1}), d(Ax_{2i-2}, Ax_{2j})\}.$$

Subcase (1.ii.1): If $d(Ax_{2j}, Bx_{2i-1}) \geq d(Ax_{2i-2}, Ax_{2j})$, we have

$$\alpha_n = d(Sx_{2i}, Ax_{2j}) \leq d(Ax_{2j}, Bx_{2i-1}),$$

which leads to the contradiction in Subcase (1.i).

Subcase (1.ii.2): Otherwise if $d(Ax_{2j}, Bx_{2i-1}) < d(Ax_{2i-2}, Ax_{2j})$, then for

k: $2i - 2 < 2k + 1 < 2j$, and for some $s, t \in \{1,2, \ldots, 5\}$, we have

$$
\begin{aligned}
\alpha_n = d(Sx_{2i}, Ax_{2j}) &\leq d(Ax_{2i-2}, Ax_{2j}) \\
&\leq d(Ax_{2i-2}, Bx_{2k+1}) + d(Ax_{2j}, Bx_{2k+1}) \\
&\leq M_\omega(x_{2i-2}, x_{2k+1}) + M_\omega(x_{2j}, x_{2k+1}) \\
&\leq \omega_s(\alpha_n) + \omega_t(\alpha_n)
\end{aligned}
$$

$$< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n$$
$$= \alpha_n,$$

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = d(Ax_{2i}, Bx_{2j+1})$ leads to a contradiction by Subcase (1.i).

Case 3: The case where $\alpha_n = d(Ax_{2i}, Ax_{2j})$ leads to a contradiction by Subcase (1.ii.2).

Case 4: If $\alpha_n = d(Bx_{2i+1}, Bx_{2j+1})$ then for $k : 2i + 1 < 2k < 2j + 1$, and for some $s, t \in \{1,2, \dots, 5\}$, we have

$$\begin{aligned}
\alpha_n &= d(Bx_{2i+1}, Bx_{2j+1}) \\
&\leq d(Ax_{2k}, Bx_{2i+1}) + d(Ax_{2k}, Bx_{2j+1}) \\
&\leq M_\omega(x_{2k}, x_{2i+1}) + M_\omega(x_{2k}, x_{2j+1}) \\
&\leq \omega_{s(\alpha_n)} + \omega_t(\alpha_n) \\
&< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \\
&= \alpha_n,
\end{aligned}$$

which is a contradiction.

Case 5: If $\alpha_n = d(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \leq i, j \leq n - 1$, then:

Subcase (5.i): If $Tx_{2i+1} = Ax_{2i}$, we have $\alpha_n = d(Tx_{2i+1}, Bx_{2j+1}) = d(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by Subcase (1.i).

Subcase (5.ii): Otherwise if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence by Lemma 1.2,

$$\alpha_n = d(Tx_{2i+1}, Bx_{2j+1}) \leq \max\{d(Bx_{2i-1}, Bx_{2j+1}), d(Ax_{2i}, Bx_{2j+1})\}.$$

This means we have either $d(Tx_{2i+1}, Bx_{2j+1}) \leq d(Bx_{2i-1}, Bx_{2j+1})$, which is a contradiction by Case 4 or $d(Tx_{2i+1}, Bx_{2j+1}) \leq d(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by Subcase (1.i).

Case 6: If $\alpha_n = d(Tx_{2i+1}, Ax_{2j})$ for some $0 \leq i, j \leq n - 1$, then:

Subcase (6.i): If $Tx_{2i+1} = Ax_{2i}$ we have $\alpha_n = d(Tx_{2i+1}, Ax_{2j}) = d(Ax_{2i}, Ax_{2j})$, which is not possible by Subcase (1.ii.2)

Subcase (6.ii): Otherwise if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence $\alpha_n = d(Tx_{2i+1}, Ax_{2j}) \leq \max\{d(Ax_{2j}, Bx_{2i-1}), d(Ax_{2i}, Ax_{2j})\}$. This implies we have either $d(Tx_{2i+1}, Ax_{2j}) \leq d(Ax_{2j}, Bx_{2i-1})$ which is a contradiction by Subcase (1.i) or else we have $d(Tx_{2i+1}, Ax_{2j}) \leq d(Ax_{2i}, Ax_{2j})$ which is a contradiction by Subcase (1.ii.2).

Case 7: If $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \leq i, j \leq n - 1$, then:

Subcase (7.i): If $Tx_{2j+1} = Ax_{2j}$, we have $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1}) = d(Tx_{2i+1}, Ax_{2j})$ which a contradiction by Case 6.

Subcase (7.ii): Otherwise if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \text{seg}[Bx_{2j-1},\ Ax_{2j}]$ and hence $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1}) \leq \max\{d(Tx_{2i+1}, Bx_{2j-1}), d(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $d(Tx_{2i+1}, Tx_{2j+1}) \leq d(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5 or else we have $d(Tx_{2i+1}, Tx_{2j+1}) \leq d(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

Case 8: If $\alpha_n = d(Sx_{2i}, Bx_{2j+1})$ for some $0 \leq i, j \leq n-1$, then:

Subcase (8.i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ then $\alpha_n = d(Sx_{2i}, Bx_{2j+1}) = d(Bx_{2i-1}, Bx_{2j+1})$, which is not possible as per Case 4. Hence $i = 0$.

Subcase (8.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it means that $Sx_{2i} \in$ seg$[Ax_{2i-2}, Bx_{2i-1}]$. This implies that
$\alpha_n = d(Sx_{2i}, Bx_{2j+1}) \leq \max\{d(Ax_{2i-1}Bx_{2j+1}), d(Bx_{2i-1}, Bx_{2j+1})\}$. This leads to a contradiction by Subcase (1.i) when
$d(Sx_{2i}, Bx_{2j+1}) \leq d(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens that
$d(Sx_{2i}, Bx_{2j+1}) \leq d(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: If $\alpha_n = d(Sx_{2i}, Sx_{2j})$ for some $0 < i < j < n-1$, then:

Subcase (9.i): If $i \geq 1$ and $Sx_{2j} = Bx_{2j-1}$, then we have
$\alpha_n = d(Sx_{2i},\ Sx_{2j}) = d(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to Case 8. Hence $i = 0$.

Subcase (9.ii): If $i \geq 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it implies that $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$ and
$d(Sx_{2i}, Sx_{2j}) \leq \max\{d(Sx_{2i}, Ax_{2j-2}),\ d(Sx_{2i}, Bx_{2j-1})\}$.
If it happens $d(Sx_{2i}, Sx_{2j}) \leq d(Sx_{2i}, Ax_{2j-2})$, we get a contradiction by Case 1. However if it happens that $d(Sx_{2i}, Sx_{2j}) \leq d(Sx_{2i}, Bx_{2j-1})$, then we get a contradiction by Case 8. Hence $i = 0$.

Case 10: If $\alpha_n = d(Sx_{2i},\ Tx_{2j+1})$, for some $0 \leq i, j \leq n-1$, we have
Subcase (10.i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ then we have
$\alpha_n = d(Sx_{2i},\ Tx_{2j+1}) = d(Tx_{2j+1}, Bx_{2i-1})$, which is not possible as per Case 8. Hence $i - 0$.
Subcase (10.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it implies that
$Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and
$d(Sx_{2i}, Tx_{2j+1}) \leq \max\{d(Tx_{2j+1}, Ax_{2i-2}), d(Tx_{2j+1}, Bx_{2i-1})\}$. This leads to contradictions by Case 6 and Case 5. Hence $i = 0$.
We have considered 10 possible cases for α_n and conclude that

$$\alpha_n \in \{d(Sx_0, Sx_{2j}), d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j+1}), d(Sx_0, Tx_{2j+1})\},$$

for some $0 \leq j \leq n-1$. By the construction of the sequences, we have
$d(Sx_0, Sx_{2j}) \leq \max\{d(Sx_0, Ax_{2j-2}), d(Sx_0, Bx_{2j-1})\}$ and
$(Sx_0, Tx_{2j+1}) \leq \max\{d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j-1})\}$. Thus we have now proved (2.3) that is,

$$\alpha_n \leq \max\{d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j+1})\}, 0 \leq j \leq 1.$$

Consider the case where $\max\{d(Sx_0, Ax_{2j})\} \le \max\{d(Sx_0, Bx_{2j+1})\}$, $0 \le j \le n-1$. Then we have for some $0 \le j \le n-1$, and for some $u \in \{1,2,\dots,5\}$

$\alpha_n \le d(Sx_0, Bx_{2j+1})$
$\quad \le d(Sx_0, Ax_0) + d(Ax_0, Bx_{2j+1})$
$\quad \le d(Sx_0, Ax_0) + \omega_u[\alpha_n]$
$\quad \le d(Sx_0, Ax_0) + 2\omega_u[\alpha_n]$
$\Rightarrow \alpha_n - 2\omega_u[\alpha_n] \le d(Sx_0, Ax_0)$.

Alternatively, if $\max\{d(Sx_0, Ax_{2j})\} > \max\{d(Sx_0, Bx_{2j+1})\}$, $0 \le j \le n-1$, then for some $0 \le j \le n-1$ and for some $v \in \{1,2,\dots,5\}$ and Subcase (1.ii.2) we have

$\alpha_n \le d(Sx_0, Ax_{2j})$
$\quad \le d(Sx_0, Ax_0) + d(Ax_0, Ax_{2j})$
$\quad \le d(Sx_0, Ax_0) + 2\omega_v[\alpha_n]$
$\Rightarrow \alpha_n - 2\omega_v[\alpha_n] \le d(Sx_0, Ax_0)$.

Thus in both cases we have for some $s \in \{1,2,3,4,5\}$

$$\alpha_n - 2\omega_s[\alpha_n] \le d(Sx_0, Ax_0) \tag{2.5}$$

By assumption (i), there is $r_0 \in [0, +\infty)$ such that for each $s \in \{1,2,\dots,5\}$, we have $r - 2\omega_s[r] > d(Sx_0, Ax_0)$ for $r > r_0$. Thus, there is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1,2,\dots,5\}$ such that for each n we have

$$a_n - 2\omega_s[a_n] \le d(Sx_0, Ax_0).$$

Thus by (2.5), $a_n \le r_0$, $n = 1,2,\dots$, and also

$$a := \lim_{n\to+\infty} a_n = diam(D) \le r_0.$$

We have hence proved that $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ are bounded sequences.

To prove that $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge in K, we reflect on the set

$$E_n = \left(\bigcup_{i=n}^{+\infty}\{Ax_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Sx_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Tx_{2i+1}\}\right),$$

$n = 2,3,\dots$.

By (2.3) we have

$$e_n := \operatorname{diam}(E_n) \le \sup_{j\ge n}\{d(Sx_{2n}, Ax_{2j}), d(Sx_{2n}, Bx_{2j+1})\}, n = 2,3,\dots$$

If $Sx_{2n} = Bx_{2n-1}$, we have, as in Case 1 and Case 8, for each $j \ge n$, and for some $u \in \{1,2,\dots,5\}$

$$e_n \le \sup_{j\ge n}\{d(Ax_{2j}, Bx_{2n-1}), d(Bx_{2j+1}, Bx_{2n-1}), d(Ax_{2j}, Bx_{2n-1}),\}, n = 2,3,\dots$$
$$\le 2\omega_u[e_{n-1}]. \tag{2.6}$$

If however $Sx_{2n} \ne Bx_{2n-1}$, it implies $Sx_{2n} \in \operatorname{seg}[Ax_{2n-2}, Bx_{2n-1}]$. Hence, as in Case 1 and Case 8, for each $j \ge n$ and for some $u \in \{1,2,\dots,5\}$, we have,

$$e_n \le \sup_{j\ge n}\{d(Ax_{2n-2}, Ax_{2j}), d(Bx_{2n-1}, Ax_{2j}), d(Ax_{2n-2}, Bx_{2j+1}), d(Bx_{2n-1}, Bx_2$$

$$\leq 2\omega_v(e_{n-2}). \tag{2.7}$$

By (2.6) and (2.7), there is a subsequence $\{\varepsilon_n\}$ of $\{e_n\}$ and some $s \in \{1,2,\ldots,5\}$ such that for each n, we have

$$\varepsilon_n \leq 2\omega_s[\varepsilon_{n-2}], n = 2,3,\ldots$$
$$\leq \varepsilon_{n-2}. \tag{2.8}$$

We note that $e_n \geq e_{n+1}$ for every n. Let $\lim_{n\to\infty} e_n = \lim_{n\to\infty} \varepsilon_n = e$. We claim that $e = 0$. If $e > 0$, then by (2.8) and assumption (i) we have

$$\lim_{n\to\infty} \varepsilon_n < \lim_{n\to\infty} \varepsilon_{n-2} \Rightarrow e < e,$$

which is a contradiction. Hence $e = 0$.

This means that the sequences $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge to a point z. Since $\{Sx_{2n}\}, \{Tx_{2n+1}\} \in K$ and $S(K), T(K)$ are closed in the complete metric space X, we conclude that

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = z \in S(K) \cap T(K). \tag{2.9}$$

As $z \in S(K)$, there is a point $u \in K$ such that $Su = z$. We show that u is a coincidence point of A, B and S.

$$\begin{aligned} d(Au, Bx_{2n+1}) &\leq \max\{\omega_1[d(Su, Tx_{2n+1})]; \omega_2[d(Au, Su)], \omega_3[d(Bx_{2n+1}, Tx_{2n+1})], \\ &\qquad \omega_4[d(Au, Tx_{2n+1})], \omega_5[d(Su, Bx_{2n+1})]\} \\ &= \max\{\omega_1[d(z, Tx_{2n+1})]; \omega_2[d(Au, z)], \omega_3[d(Bx_{2n+1}, Tx_{2n+1})], \\ &\qquad \omega_4[d(Au, Tx_{2n+1})], \omega_5[d(z, Bx_{2n+1})]\}. \end{aligned}$$

Taking $n \to +\infty$ and applying (2.9) we get

$$\begin{aligned} d(Au, z) &\leq \max\{\omega_1[d(z,z)], \omega_2[d(Au,z)], \omega_3[d(z,z], \omega_4[d(Au,z)], \omega_5[d(z,z)]\} \\ &\leq \omega_i[d(Au,z)] \text{ for some } i \in \{2,4\} \\ &< d(Au,z) \text{ for } d(Au,z) > 0 \\ &\Rightarrow d(Au,z) = 0 \end{aligned}$$
$$\Rightarrow Au = z.$$

Using a similar procedure, when we expand $d(Ax_{2n}, Bu)$, we get $Bu = z$ making u a coincidence point of A, B and S. By the coincidental commutativity of S and A we have

$$SAu = ASu \Rightarrow Sz = Az.$$

From (2.9), $z \in T(K)$ means there is $v \in K$, such that $Tv = z$. We show that $Bv = z$.

$$\begin{aligned} d(z, Bv) &= d(Au, Bv) \\ &\leq \max\{\omega_1[d(Su, Tv)], \omega_2[d(Su, Au)], \omega_3[d(Tv, Bv)], \\ &\qquad \omega_4[d(Au, Tv)], \omega_5[d(Su, Bv)]\} \\ &= \max\{\omega_1[d(z,z)], \omega_2[d(z,z)], \omega_3[d(z, Bv)], \omega_4[d(z,z)], \omega_5[d(z, Bv)]\} \\ &\leq \omega_j[d(z, Bv)] \text{ for } j = 3 \text{ or } 5, \\ &< d(z, Bv) \text{ for } d(z, Bv) > 0 \\ &\Rightarrow Bv = z. \end{aligned}$$

Thus v is a coincidence point of B and T. By the coincidental commutativity property, we have $BTv = TBv \Rightarrow Bz = Tz$.

$$\begin{aligned} d(Az, Bz) &\leq \max\{\omega_1[d(Sz, Tz)], \omega_2[d(Sz, Az)], \omega_3[d(Tz, Bz)], \\ &\qquad \omega_4[d(Az, Tz)], \omega_5[d(Sz, Bz)]\} \\ &= \max\{\omega_1[d(Az, Bz)], \omega_2[d(Az, Az)], \omega_3[d(Bz, Bz)], \end{aligned}$$

$$\omega_4[d(Az,Bz)],\omega_5[d(Az,Bz)]\}$$
$$\leq \omega_i[d(Az,Bz)] \text{ for } i \in \{1,4,5\}$$
$$< d(Az,Bz) \text{ for } d(Az,Bz) > 0$$
$$\Rightarrow Az = Bz.$$

Hence we have

$$Az = Bz = Sz = Tz. \tag{2.10}$$

Now we consider the following:

$$\begin{aligned} d(z,Bz) &= d(Au,Bz) \\ &\leq \max\{\omega_1[d(Su,Tz)],\omega_2[d(Au,Su)],\omega_3[d(Bz,Tz)], \\ &\qquad \omega_4[d(Au,Tz)],\omega_5[d(Su,Bz)]\} \\ &\leq \max\{\omega_1[d(z,Bz)],\omega_2[d(z,z)],\omega_3[d(Bz,Bz)], \\ &\qquad \omega_4[d(z,Bz)],\omega_5[d(z,Bz)]\} \\ &\leq \omega_j[d(z,Bz)] \text{ for } j \in \{1,4,5\} \\ &< d(z,Bz) \text{ for } d(z,Bz) > 0 \\ \Rightarrow d(z,Bz) &= 0 \\ \Rightarrow Bz &= z. \end{aligned}$$

From (2.10) we conclude that

$$Az = Bz = Sz = Tz = z.$$

This means that z is a common fixed point of A, B, S and T.

We now show that z is unique. Suppose z' is also a common fixed point of A, B, S and T. We get

$$\begin{aligned} d(z,z') &= d(Az,Bz') \\ &\leq \max\{\omega_1[d(Sz,Tz')],\ \omega_2[d(Az,Sz)],\ \omega_3[d(Bz',Tz')], \\ &\qquad \omega_4[d(Az,Tz')],\omega_5[d(Sz,Bz')]\} \\ &\leq \max\{\omega_1[d(z,z')],\ \omega_2[d(z,z)],\ \omega_3[d(z',z')],\omega_4[d(z,z')],\omega_5[d(z,z')]\} \\ &\leq \omega_k[d(z,z')] \text{ for } k \in \{1,4,5\} \\ &< d(z,z') \text{ for } d(z,z') > 0 \\ &\Rightarrow d(z,z') = 0 \\ &\Rightarrow z = z'. \end{aligned}$$

This proves that the common fixed point of A, B, S and T is unique.

If we define $\omega_t[r] = hr$ for $0 \leq 2h < 1$, we get the following corollary:

Corollary 2.2. *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X and δK be the boundary of K, with $\delta K \neq \emptyset$. Let the mappings $A, B, S, T: K \to X$. Suppose that A, B, S and T satisfy the following conditions:*

(i) *For every $x, y \in K$ we have $d(Ax,By) \leq hM(x,y)$, where $0 \leq 2h < 1$ and $M(x,y) = max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Ax,T y), d(Sx,By)\}$,*
(ii) $\delta K \subseteq T(K), \delta K \subseteq S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K, Tx \in \delta K \Rightarrow Bx \in K$ *and*
(iv) $S(K), T(K)$ *are closed in* X.

Then there exists a coincidence point z for A, B, S and T in K. Moreover, if each of the pairs $\{A, S\}$ and $\{B, T\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

We deduce another corollary by letting $A = B$. When this is the situation, in the proof for Theorem 2.1, Case 4 is identical to Subcase (1.i). Moreover, Subcase (1.i) enables us to change the property in Theorem 2.1(i) from $\omega_1[r] < r/2$ for $r > 0$ to $\omega_1[r] < r$ for $r > 0$.

Corollary 2.3. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X and δK be the boundary of K, with $\delta K \neq \emptyset$. Let the mappings $A, S, T: K \to X$. Suppose that A, S and T satisfy the following conditions:*

(i) For every $x, y \in K, d(Ax, Ay) \leq M_\omega(x, y)$ where
$M_\omega(x, y) =$
$max\{\omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(Ay, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, Ay)]\}$ *where $\omega_1: [0, +\infty) \to [0, +\infty), i = 1,2,3,4,5,$ is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $lim_{r\to\infty}[r - w_i(r)] = +\infty$,*
(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Ax \in K$,
(iv) $A(K) \cap K \subset T(K), A(K) \cap K \subset S(K)$ and
(v) $S(K), T(K)$ is closed in X.
Then there exists a coincidence point $z \in K$ for A, S and T. Moreover, if each of the pairs $\{A, S\}$ and $\{A, T\}$ is coincidentally commuting, then z remains a unique common fixed point of A, S and T.
Remark 1: If we set $S = T$ in Corollary 2.3, we get Theorem 1.3 by Gajić and Rakočević [1].

Remark 2: If we set $S = T = I$ in Corollary 2.3, we get the theorem as proved by Ćirić [5].

We form the following corollary by setting $A = B = I$ in Theorem 2.1, that is, setting $A = I$ in Corollary 2.3.

Corollary 2.4. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X and δK be the boundary of K, with $\delta K \neq \emptyset$. Let the mappings $S, T: K \to X$. Suppose that S and T satisfy the following conditions:*
(i) For every $x, y \in K, d(x, y) \leq M_\omega(x, y)$ where
$M_\omega(x, y) = \max\{\omega_1[d(Sx, Ty)], \omega_2[d(x, Sx)], \omega_3[d(y, Ty)], \omega_4[d(x, Ty)], \omega_5[d(Sx, y)]\}$ and $\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5,$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $lim_{r\to\infty}[r - w_i(r)] = +\infty$,*
(ii) $Sx \in \delta K \Rightarrow x \in K,\ Tx \in \delta K \Rightarrow x \in K$,
(iii) $K \subset T(K), K \subset S(K)$ and
(iv) $S(K), T(K)$ is closed in X.
Then there exists a unique common fixed point of S and T.

We form yet another corollary from Corollary 2.4 by setting $S = T$.

Corollary 2.5. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X and δK*

be the boundary of K, with $\delta K \neq \emptyset$. Let the mapping $S: K \to X$. Suppose that S satisfies the following conditions:

(i) For every $x, y \in K, d(x,y) \leq M_{\omega}(x,y)$ where
$M_{\omega}(x,y) = \max\{\omega_1[d(Sx,Sy)], \omega_2[d(x,Sx)], \omega_3[d(y,Sy)], \omega_4[d(x,Sy)], \omega_5[d(Sx,y)]\}$ and $\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $lim_{r\to\infty}[r - w_i(r)] = +\infty$,*

(ii) $Sx \in \delta K \Rightarrow x \in K$,

(iii) $K \subset S(K)$ and

(iv) $S(K)$ is closed in X.

Then there exists a unique fixed point of S.

If we let $B = I$ in Theorem 2.1, to get the following corollary:

Corollary 2.6.
Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $A, S, T : K \to X$. Assume that A, S and T satisfy the following conditions:

(vi) For every $x, y \in K, d(Ax,y) \leq M_{\omega}(x,y)$, where
$M_{\omega}(x,y)$
$= max\{[\omega_1[d(Sx,Ty)], \omega_2[d(Ax,Sx)], \omega_3[d(y,Ty)], \omega_4[d(Ax,Ty)], \omega_5[d(Sx,y)]\}$,
$\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$,*

(vii) $\delta K \subseteq T(K)$,

(viii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,

(ix) $A(K) \cap K \subset T(K), K \subset S(K)$ and

(x) $S(K), T(K)$ are closed in X.

Then there exists a coincidence point $z \in K$ for A, S and T. Moreover, if the pair $\{A, S\}$ is coincidentally commuting, then z remains a unique common fixed point of A, S and T.

When we set S = T in Theorem 2.1, we get the following corollary:

Corollary 2.7. *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $A, B, S : K \to X$. Assume that A, B and S satisfy the following conditions:*

(xi) For every $x, y \in K, d(Ax,By) \leq M_{\omega}(x,y)$, where
$M_{\omega}(x,y)$
$= max\{[\omega_1[d(Sx,Sy)], \omega_2[d(Ax,Sx)], \omega_3[d(By,Sy)], \omega_4[d(Ax,Sy)], \omega_5[d(Sx,By)]\}$,
$\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$.*

(xii) $\delta K \subset S(K)$,

(xiii) $Sx \in \delta K \Rightarrow Ax, Bx \in K$,

(xiv) $A(K) \cap K \subset S(K), B(K) \cap K \subset S(K)$ and

(xv) $S(K)$, is closed in X.

Then there exists a coincidence point $z \in K$ for A, B and S. Moreover, if each of the pairs $\{S, A\}$ and $\{S, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B and S.

We form another corollary by setting $x = y$ in Theorem 2.1.

Corollary 2.8. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $S, T, A, B : K \to X$. Assume that S, T, A and B satisfy the following conditions:*

(i) *For every $x, y \in K, d(Ax, Bx) \leq M_\omega(x)$, where*
$M_\omega(x)$
$= max\{[\omega_1[d(Sx, Tx)], \omega_2[d(Ax, Sx)], \omega_3[d(Bx, Tx)], \omega_4[d(Ax, Tx)], \omega_5[d(Sx, Bx)]\}$,
$\omega_1 : [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5,$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2} r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$.*
(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,
(iv) $A(K) \cap K \subset T(K), B(K) \cap K \subset S(K)$ *and*
(v) $S(K), T(K)$ *is closed in X.*
Then there exists a coincidence point $z \in K$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Here we give an example on the use of our result (Theorem 2.1).

Example 2.1: Let $X = [0, +\infty), K = [1, 3]$ and $d(x, y) = |x - y|$.
Let $\omega_i[r] = \frac{1}{3} r$ for $i \in \{1, 2, 3, 4, 5\}$. We note that $\omega_i[r] < \frac{1}{2} r$. Define $A, B, S, T : K \to X$ by

$$Sx = \begin{cases} 2x^4 - 1 & for\ x \in [1,2] \\ 7 & for\ x \in (2,3] \end{cases}, \qquad Ax = \begin{cases} x^2 & for\ x \in [1,2] \\ 1 & for\ x \in (2,3] \end{cases}$$

$$Sx = \begin{cases} 2x^6 - 1 & for\ x \in [1,2] \\ 7 & for\ x \in (2,3] \end{cases}, \qquad Ax = \begin{cases} x^3 & for\ x \in [1,2] \\ 2 & for\ x \in (2,3] \end{cases}.$$

We have $S(K) = [1, 31], T(K) = [0, 127]$ both of which are closed. We also have $\delta K = \{1,3\} \subseteq S(K), T(K)$.
We find out that $\{x \in K : Sx \in \delta K\} = \{1, 2^{1/4}\}$ and $A(\{1, 2^{1/4}\}) = \{1, \sqrt{2}\} \in K$. Similarly $\{x \in K : Tx \in \delta K\} = \{1, 2^{1/6}\}$ and $B(\{1, 2^{1/6}\}) = \{1, \sqrt{2}\} \in K$.
We note that $\{A, S\}$ and $\{B, T\}$ are both coincidentally commuting at $x = 1$, that is, $SA(1) = AS(1) = 1$ and $TB(1) = BT(1) = 1$. We also note that all four mappings are discontinuous at $x = 2$. Without loss of generality let $y \geq x$.

Consider $x, y \in (2, 3]$. Then we have

$$d(Ax, By) = d(2,2) = 0 \leq \frac{1}{3} d(Sx, Ty).$$

For $x, y \in [1,2]$, we have

$$
\begin{aligned}
d(Ax,By) &= |x^2 - y^3| \\
&= |x^2 - y^3| \times \frac{|x^2 + y^3|}{|x^2 + y^3|} \\
&= \frac{|x^4 - y^6|}{|x^2 + y^3|} \\
&\leq \frac{1}{2} \times \frac{1}{2} |2x^4 - 2x^6| \\
&= \frac{1}{4} d(Sx,Ty) \\
&\leq \frac{1}{3} d(Sx,Ty).
\end{aligned}
$$

Finally, for $x \in [1,2]$, $y \in (2,3]$, we get

$$
\begin{aligned}
d(Ax,By) &= |x^2 - 2| \\
&= |x^2 - 2| \frac{|x^2 + 2|}{|x^2 + 2|} \\
&= \frac{|x^4 - 4|}{|x^2 + 2|} \\
&< \frac{1}{3} \times \frac{1}{2} |2x^4 - 8| \\
&= \frac{1}{6} d(Sx,Ty) \\
&\leq \frac{1}{3} d(Sx,Ty).
\end{aligned}
$$

Thus in all cases, for every $x, y \in K$, we have

$$
d(Ax,By) \leq \max\{ \omega_1[d(Sx,Ty)], \omega_2[d(Ax,Sx)], \omega_3[d(By,Ty)], \omega_4[d(Ax,Ty)], \omega_5[d(Sx,By)]\}.
$$

Thus all the conditions of Theorem 2.1 are satisfied and 1 is the unique common fixed point of A, B, S and T.

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