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COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In many situations the arising mappings need not be self-mappings. It is well known that metrically convex metric spaces are desirable for the study of fixed points of non-self mappings. Partial metric spaces are generalization of the notion of a metric space that allows non-zero self distance. In this paper, we present a common fixed point theorem for weakly compatible mappings on complete metrically convex partial metric spaces. Our main result generalizes a common metric fixed point theorem due to Ćirić and Ume, which is an extension of a metric fixed point theorem due to Rhoades, to partial metric spaces.

1. Introduction and Preliminaries

A metric fixed point theorem is an extension in some sense of the well known Banach's fixed point theorem or the Banach contraction principle. Jungck [3] introduced and studied the notion of compatible mappings, and proved a common fixed point theorem for the mappings on complete metric spaces. Nadler [14] initiated the study of fixed points for multi-valued contraction mappings on metric spaces, and proved the version of the Banach contraction principle for the mappings. Kaneko and Sessa [5] extended the notion of compatible mappings due to Jungck [3] to include multi-valued

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mappings, and proved some metric fixed point theorems for single-valued and multi-valued mappings.

Assad and Kirk [11] initiated the study of fixed points for non-self mappings on metric spaces, and proved fixed point theorems for single-valued and multi-valued contraction mappings on complete metrically convex metric spaces. Rhoades [2] introduced a class of single-valued generalized contraction non-self mappings, and proved a fixed point theorem for the class of mappings, which generalizes the theorem due to Assad and Kirk [11]. The results due to Rhoades [2] were extended and improved by Ćirić [7]. Rhoades [1] proved a fixed point theorem for multi-valued generalized contraction non-self mappings, a generalization of the fixed point theorems due to Ćirić [7] and Rhoades [2].

Ahmed and Rhoades [9] extended the notion of compatible mappings due to Jungck [3], and proved some common fixed point theorems for compatible single-valued and multi-valued non-self mappings. Ćirić and Ume [8] further extended the notion of compatible non-self mappings to weakly compatible non-self mappings, and they used improved versions of the methods by Ahmed and Rhoades [9] and Rhoades [1], to established a common fixed point theorem, an extension of the theorem due to Rhoades [1] for two multi-valued and one single-valued weakly compatible mappings.

Partial metric spaces were introduced by Matthews [12] while studying denotational semantics of data flow networks, showing that the Banach's contraction principle generalizes to partial metric spaces. Partial metrics have many applications; In particular, they are desirable in modeling of partially defined information, often encountered in computer sciences [10]. Matthews [13] studied topological aspects for partial metric spaces. Aydi, Abbas and Vetro [4] initiated the study of fixed points for multi-valued contraction mappings on partial metric spaces; They introduced and studied the notion of *partial Hausdorff metric*, and obtain a generalization of the Nadler's fixed point theorem for multi-valued contraction mappings [14] to partial metric spaces.

Recently, there has been several studies on possible generalizations of the existing metric fixed point results to partial metric spaces. This paper forms a part of the studies for metric fixed point results for weakly compatible mappings.

The purpose of this paper is to generalize a metric fixed point theorem due to Ćirić and Ume [8] to partial metric spaces.

We will require the following definitions and preliminary results to prove our results.

Definition 1.1. Let X be a non-empty set. Let $T : X \to 2^X$, where 2^X denotes the collection of all non-empty subsets of X, be a multi-valued mapping and $f : X \to X$ be a single-valued mapping, then a point $t \in X$ is called a common fixed point of T and f if $t = ft \in Tt$.

Definition 1.2 ([10]). Let X be non-empty set. A partial metric space is a pair (X, p), where p is a function $p : X \times X \to \mathbb{R}^+$, called the partial metric, such that for all x, y, $z \in X$:

(P1) $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$; and
(P4) $p(x, y) + p(z, z) \leq p(x, z) + p(z, y)$.

Clearly, by (P1) - (P3), $p(x, y) = 0$ implies $x = y$. But, the converse is in general not true.

A basic example of partial metric space is the pair $([0, \infty), p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. More examples of partial metric spaces may be found in [13, 10].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and ε is a real number.

Let (X, p) be a partial metric space, A any non-empty subset of the set X and x an element of the set X. It is well known [15] that $x \in \bar{A}$, where $\bar{A}$ is the closure of A, if and only if $p(x, A) = p(x, x)$. Also, the set A is said to be closed in (X, p) if and only if $A = \bar{A}$.

Definition 1.3 ([10]). Let (X, p) be a partial metric space. Then:

(i) A sequence $\{x_n\}$ in (X, p) is said to converge to some $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to the topology τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1.4 ([10]). *Let (X, p) be a partial metric space. Then the mapping $p^s : X \times X \to [0, \infty)$ given by*

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

for all $x, y \in X$ defines a metric on X.

Lemma 1.5 ([10]). *Let (X, p) be a partial metric space. Then:*

(i) *A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*
(ii) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

Let (X, d) be a metric space and $CB(X)$ denotes the collection of all non-empty bounded closed subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance from a point $x \in X$ to the set $A \in CB(X)$. It is well known [14] that H is a metric, called the Pompeiu-Hausdorff metric, on $CB(X)$ induced by the metric d. The metric space $(CB(X), H)$ is complete whenever (X, d) is complete.

Definition 1.6 ([4])**.** Let (X, p) be a partial metric space and $CB^p(X)$ denotes the collection of all non-empty bounded and closed subsets of X. For A, $B \in CB^p(X)$ and $x \in X$, define

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}$$

where $p(x, A) = \inf\{p(x, a) : a \in A\}$, $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$ and $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$. Then the mapping H_p is a partial metric, called the *partial Hausdorff metric*, on $CB^p(X)$ induced by the partial metric p.

Lemma 1.7 ([4])**.** *Let (X, p) be a partial metric space. Let A, $B \in CB^p(X)$ and $q > 1$. Then for any $a \in A$, there exists $b \in B$ that depends on a such that*

$$p(a, b) \leq qH_p(A, B).$$

Pathak [6] introduced and studied the notion of f-weak compatible mappings, and proved coincidence and metric fixed point theorems for single-valued and multi-valued f-weak compatible mappings, an extension of theorems due to Kaneko and Sessa [5].

Definition 1.8 ([6])**.** Let (X, d) be a metric space and $F : X \to CB(X)$ and $f : X \to X$ be mappings. The mappings F and f are f-weak compatible if $fFx \in CB(X)$ for all $x \in X$ and the following limits exist and satisfy:

(i) $\lim_{n\to\infty} H(fFx_n, Ffx_n) \leq \lim_{n\to\infty} H(Ffx_n, Fx_n)$, and

(ii) $\lim_{n\to\infty} d(fFx_n, fx_n) \leq \lim_{n\to\infty} H(Ffx_n, Fx_n)$, whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to A \in CB(X)$ and $fx_n \to s \in A$.

Example 1 in Pathak [6] shows that compatible mappings F and f are f-weak compatible mappings, but the converse may not hold.

Ćirić and Ume [8] extended the notion of f-weak compatible mappings to weakly compatible mappings, which drop the following strict requirements of Definition 1.8 namely: $fFx \in CB(X)$ for all $x \in X$, $A \in CB(X)$ and f and F must be self-mappings.

Definition 1.9 ([8])**.** Let (X, d) be a metric space and K be a non-empty subset of X. The mappings $F : K \to CB(X)$ and $f : K \to X$ are said to be weakly compatible on K if, for any sequences $\{x_n\}$ and $\{y_n\}$ in K such that $fx_n \in K$, $Fx_n \cap K \neq \varnothing$, the following limits exist and satisfy:

(i) $\limsup_{n\to\infty} D(fy_n, Ffx_n) \leq \limsup_{n\to\infty} H(Ffx_n, Fx_n)$, and
(ii) $\limsup_{n\to\infty} d(fy_n, fx_n) \leq \limsup_{n\to\infty} H(Ffx_n, Fx_n)$, whenever $\{y_n\} \in Fx_n \cap K$ and $\lim_{n\to\infty} d(y_n, fx_n) = 0$.

Definition 1.10 ([11])**.** Let (X, d) be a metric space. If for each $x, y \in X$, $x \neq y$ there exists $z \in X$, $x \neq y \neq z$, such that $d(x, z) + d(z, y) = d(x, y)$, then X is said to be a metrically convex metric space.

Lemma 1.11 ([11])**.** *Let (X, d) be a metrically convex metric space and K a non-empty closed subset of X. Then for given $x \in K$, $y \notin K$, there exists $z \in \partial K$, boundary of K, such $d(x, z) + d(z, y) = d(x, y)$.*

Analogous to Definition 1.10 we have the following definition.

Definition 1.12. Let (X, p) be a partial metric space. If for each x, $y \in X$, $x \neq y$ there exists $z \in X$, $x \neq y \neq z$ such that $p(x, z) + p(z, y) = p(x, y) + p(z, z)$, then X is said to be a metrically convex partial metric space.

Example 1.13. *Let $X = [0, \infty)$ be equipped with a partial metric $p(x, y) = \max\{x, y\}$ for all x, $y \in X$. Clearly, (X, p) is a metrically convex partial metric space. Observe that for any x, $y \in X$, there is a $z \in X$ with $x < z < y$ such that $p(x, z) + p(z, y) = p(x, y) + p(z, z)$.*

Remark 1.14. Note that when $p = p^s$ in Definition 1.12, we have $p^s(x, y) = p^s(x, z) + p^s(z, y)$. Therefore, if (X, p) is a metrically convex partial metric space, then (X, p^s) is a metrically convex metric space. Also, by Lemma 1.4 $p^s(x, y) = p^s(x, z) + p^s(x, z)$ simplifies to $p(x, y) + p(z, z) = p(x, z) + p(z, y)$.

The following result, a generalization of Lemma 1.11 to partial metric spaces, is immediate.

Lemma 1.15. *Let (X, p) be a metrically convex partial metric space and K a non-empty closed subset of X. Then given $x \in K$, $y \notin K$, there exists $z \in \partial K$, boundary of K, such $p(x, z) + p(z, y) = p(x, y) + p(z, z)$.*

Definition 1.16. Let (X, p) be a partial metric space and K be a non-empty subset of X. Let $F : K \to CB^p(X)$ and $f : K \to X$ be mappings. If for any sequences $\{x_n\}$ and $\{y_n\}$ in K such that $fx_n \in K$, $Fx_n \cap K \neq \varnothing$, the following hold:

(i) $\limsup_{n\to\infty} p(fy_n, Ffx_n) \leq \limsup_{n\to\infty} H_p(Ffx_n, Fx_n)$, and
(ii) $\limsup_{n\to\infty} p(fy_n, fx_n) \leq \limsup_{n\to\infty} H_p(Ffx_n, Fx_n)$, whenever $\{y_n\} \in Fx_n \cap K$ and $\lim_{n\to\infty} p(y_n, fx_n) = 0$.

Then, the mappings $\{F, f\}$ are said to be weakly compatible on K.

Ćirić and Ume [8] established the following common fixed point theorem.

Theorem 1.17. *Let (X, d) be a complete and metrically convex metric space and K be a non-empty closed subset of X. Let the mappings F, $G : K \to CB(X)$ and $f : K \to X$ satisfy the following condition:*

$$H(Fx, Gy) \leq h \max \left\{ \frac{d(fx, fy)}{a}, D(fx, Fx), D(fy, Gy), \frac{D(fx, Gy) + D(fy, Fx)}{a + h} \right\} \tag{1.1}$$

for all $x, y \in X$, where $0 < h < \frac{2}{3}$, $a \geq 1 + \frac{2h^2}{1+h}$, and

(i) *$\partial K \subseteq fK$, $FK \cap K \subseteq fK$, $GK \cap K \subseteq fK$,*
(ii) *$fx \in \partial K \Rightarrow Fx \subseteq K$ and $Gx \subseteq K$,*
(iii) *$\{F, f\}$ and $\{G, f\}$ are weakly compatible mappings, and*
(iv) *f is continuous on K.*

Then, there exists a point s in K such that $s = fs \in Fs \cap Gs$.

In this paper Theorem 1.17 is generalized to partial metric spaces.

2. Main results

The following theorem is a generalization of Theorem 1.17 to partial metric spaces.

Theorem 2.1. *Let (X, p) be a metrically convex complete partial metric space. Let K be a non-empty closed subset of X. Let the mappings F, $G : K \to CB^p(X)$ and $f : K \to X$ satisfy the following condition:*

$$H(Fx, Gy) \leq h \max \left\{ \frac{p(fx, fy)}{a}, p(fx, Fx), p(fy, Gy), \frac{p(fx, Gy) + p(fy, Fx)}{a + h} \right\} \tag{2.1}$$

for all $x, y \in X$, where $0 < h < \frac{2}{3}$, $a \geq 1 + \frac{2h^2}{1+h}$, and

(i) *$\partial K \subseteq fK$, $FK \cap K \subseteq fK$, $GK \cap K \subseteq fK$,*
(ii) *$fx \in \partial K \Rightarrow Fx \subseteq K$ and $Gx \subseteq K$,*
(iii) *$\{F, f\}$ and $\{G, f\}$ are weakly compatible mappings, and*
(iv) *f is continuous on K.*

Then, the mappings f, F and G have a common fixed point. i.e, there exists a point s in K such that $s = fs \in Fs \cap Gs$.

Proof. We will construct three sequences: $\{x_n\} \subseteq K$, $\{y_n\} \subseteq FK \cup GK$, and $\{z_n\} \subseteq fK \cap K$ in the following way: Let z_0 be an arbitrary point of ∂K. Since $z_0 \in \partial K$ and $\partial K \subseteq fK$, we can find a point $x_0 \in K$ such that $fx_0 = z_0 \in \partial K$. By Theorem 2.1(ii) we have $Fx_0 \subseteq K$, and by Theorem 2.1 (i) we can find a point $x_1 \in K$ such that $fx_1 \in Fx_0 \subseteq K$. Set $y_1 = z_1 = fx_1$. Let q be any real number such that

$$q > 1 \quad \text{and} \quad qh = t < \frac{2}{3}. \tag{2.2}$$

Since $y_1 \in Fx_0$, by Lemma 1.7, there exists a point $y_2 \in Gx_1$ such that

$$p(y_1, y_2) \leq qH_p(Fx_0, Gx_1).$$

Clearly, y_2 may or may not be a point in K. If $y_2 \in K$ then by Theorem 2.1 (i), there exists a point $x_2 \in K$ such that $fx_2 = z_2 = y_2 \in Gx_1$. If $y_2 \notin K$, Using Lemma 1.15, we set $z_2 \in \partial K$ such that $p(y_1, z_2) + p(z_2, y_2) = p(y_1, y_2) + p(z_2, z_2)$. Since $z_2 \in \partial K$ and $\partial K \subseteq fK$, there exists $x_2 \in K$ such that $fx_2 = z_2 \in \partial K$.

Two possibilities to consider:

$$p(z_2, y_2) \leq \frac{1}{2}[p(y_1, y_2) + p(z_2, z_2)] \text{ or } p(y_1, z_2) < \frac{1}{2}[p(y_1, y_2) + p(z_2, z_2)].$$

For $p(z_2, y_2) \leq \frac{1}{2}[p(y_1, y_2) + p(z_2, z_2)]$, choose a point $y_3 \in Fx_2 \subseteq K$ such that $p(y_2, y_3) \leq qH_p(Gx_1, Fx_2)$. Since $Fx_2 \subseteq FK \cap K$, by Theorem 2.1(i) we can find $x_3 \in K$ such that $fx_3 = y_3 \in Fx_2 \subseteq K$.

For $p(y_1, z_2) < \frac{1}{2}[p(y_1, y_2) + p(z_2, z_2)]$. As $y_1 \in Fx_0$, Choose $y_3 \in Gx_2 \subseteq K$ such that $p(y_1, y_3) \leq qH_p(Fx_0, Gx_1)$. Since $y_3 \in GK \cap K \subseteq fK$, we can find a point $x_3 \in K$ such that $fx_3 = y_3 \in Gx_2 \subseteq K$.

In general, we construct the sequences $\{x_n\} \subseteq K$, $\{y_n\} \subseteq FK \cup GK$, and $\{z_n\} \subseteq fK \cap K$ such that:

(i) $y_n \in Fx_{n-1}$, or $y_n \in Gx_{n-1}$,
(ii) $z_n = fx_n \in K$,
(iii) $y_n = z_n$ if and only if $y_n \in K$, and in this case y_{n+1} satisfies the following condition: $y_{n+1} \in Fx_n$ and $p(y_n, y_{n+1}) \leq qH_p(Gx_{n-1}, Fx_n)$, if $y_n \in Gx_{n-1}$, or
$y_{n+1} \in Gx_n$ and $p(y_n, y_{n+1}) \leq qH_p(Fx_{n-1}, Gx_n)$, if $y_n \in Fx_{n-1}$.
(iv) $y_n \neq z_n$ whenever $y_n \notin K$, when $z_n \in \partial K$ satisfies $p(y_{n-1}, z_n) + p(z_n, y_n) = p(y_{n-1}, y_n) + p(z_n, z_n)$.

There are two possibilities to consider:

(iva) if $p(z_n, y_n) \leq \frac{1}{2}p(y_{n-1}, y_n)$
if $y_n \in Gx_{n-1}$, then $y_{n+1} \in Fx_n$ and $p(y_n, y_{n+1}) \leq qH_p(Gx_{n-1}, Fx_n)$
or
if $y_n \in Fx_{n-1}$, then $y_{n+1} \in Gx_n$ and $p(y_n, y_{n+1}) \leq qH_p(Fx_{n-1}, Gx_n)$.
(ivb) $p(y_{n-1}, z_n) < \frac{1}{2}[p(y_{n-1}, y_n) + p(z_n, z_n)]$, then:
if $y_n \in Fx_{n-1}$, then $y_{n-1} \in Gx_{n-2}$ and $y_{n+1} \in Fx_n$ is such that $p(y_{n-1}, y_{n+1}) \leq qH_p(Gx_{n-2}, Fx_n)$, or
if $y_n \in Gx_{n-1}$, then $y_{n-1} \in Fx_{n-2}$ and $y_{n+1} \in Gx_n$ is such that $p(y_{n-1}, y_{n+1}) \leq qH_p(Fx_{n-2}, Gx_n)$.

Remark 2.2. We note that if $z_n \neq y_n$ (when $z_n \in \partial K$ and $z_n \notin FK \cup GK$), then $z_{n+1} = y_{n+1} \in K$ and $z_{n-1} = y_{n-1} \in K$.

We now estimate $p(z_n, z_{n+1})$.

By construction of $\{z_n\}$, clearly, if $p(z_n, z_{n+1}) = 0$ for some n, then $z_{n+k} = z_n$ for all $k \geq 1$.

Now suppose that $p(z_n, z_{n+1}) > 0$ for all n. By Remark 2.2 we have three possibilities to consider.

Case 1. *Let $z_n = y_n \in K$ and $z_{n+1} = y_{n+1} \in K$. Suppose, without loss of generality, that $z_n = y_n = fx_n \in Fx_{n-1}$. Then $z_{n+1} = y_{n+1} \in Gx_n$, $z_{n-1} = fx_{n-1}$, and y_n and y_{n+1} satisfy $p(y_n, y_{n+1}) \leq qH_p(Fx_{n-1}, Gx_n)$. By (2.1) and (2.2) we have the following:*

$$\begin{aligned} p(y_n, y_{n+1}) \leq\ & qH_p(Fx_{n-1}, Gx_n) \\ \leq\ & t \max \left\{ \frac{p(fx_{n-1}, fx_n)}{a}, p(fx_{n-1}, Fx_{n-1}), p(fx_n, Gx_n), \right. \\ & \left. \frac{p(fx_{n-1}, Gx_n) + p(fx_n, Fx_{n-1})}{a+h} \right\} \\ \leq\ & t \max \left\{ \frac{p(z_{n-1}, z_n)}{a}, p(z_{n-1}, z_n), p(z_n, z_{n+1}), \right. \\ & \left. \frac{p(z_{n-1}, z_{n+1}) + p(z_n, z_n)}{a+h} \right\} \end{aligned} \tag{2.3}$$

Clearly if (2.1) holds for some $h_1 > 0$, then it holds for any h with $h_1 \leq h < \frac{2}{3}$; and, in particular, for $\frac{3}{5} \leq h < \frac{2}{3} \Rightarrow \frac{9}{20} \leq \frac{2h^2}{1+h} < \frac{8}{15}$. Then $\frac{1}{a} < \frac{3}{4}$ and $\frac{1}{a+h} < \frac{1}{2}$. Using (2.1) we have,

$$\begin{aligned} H_p(Fx, Gy) \leq\ & h \max \left\{ \frac{3p(fx, fy)}{4}, p(fx, Fx), p(fy, Gy), \right. \\ & \left. \frac{p(fx, Gy) + p(fy, Fx)}{2} \right\} \end{aligned} \tag{2.4}$$

From (2.3) and by Definition 1.2 (P4) we have,

$$\begin{aligned} p(y_n, y_{n+1}) \leq\ & t \max \left\{ p(z_{n-1}, z_n), p(z_n, z_{n+1}), \frac{p(z_{n-1}, z_n) + p(z_n, z_{n+1})}{2} \right\} \\ \leq\ & t \max\{p(z_{n-1}, z_n), p(z_n, z_{n+1})\}. \end{aligned}$$

Hence, as $z_n = y_n$, $z_{n+1} = y_{n+1}$ and $t < \frac{2}{3}$, We necessarily have

$$p(y_n, y_{n+1}) \leq tp(z_{n-1}, z_n). \tag{2.5}$$

which follows

$$p(z_n, z_{n+1}) \leq tp(z_{n-1}, z_n). \tag{2.6}$$

We notice that (2.5) and hence (2.6) hold, whenever $y_n = z_n$, regardless of whether $y_{n+1} = z_{n+1}$ or not.

Case 2. *Suppose* $z_n = y_n \in K$, *but* $z_{n+1} \neq y_{n+1} \notin K$. *Then* $z_{n+1} \in \partial K$ *is such that*

$$p(y_n, z_{n+1}) + p(z_{n+1}, y_{n+1}) = p(y_n, y_{n+1}) + p(z_{n+1}, z_{n+1}).$$

This implies that

$$p(y_n, z_{n+1}) < p(y_n, y_{n+1})$$

since

$$p(z_{n+1}, z_{n+1}) \leq p(z_{n+1}, y_{n+1}) \text{ (see Definition 1.2).}$$

By implication of (2.5) *we conclude*

$$p(z_n, z_{n+1}) \leq tp(z_{n-1}, z_n). \tag{2.7}$$

Case 3. *Suppose* $z_n \neq y_n$. *Then* $z_n \in \partial K$, $z_{n+1} = y_{n+1}$, $z_{n-1} = y_{n-1}$, *and*

$$p(y_{n-1}, z_n) + p(z_n, y_n) = p(y_{n-1}, y_n) + p(z_n, z_n). \tag{2.8}$$

Assume that $p(z_n, z_{n+1}) \leq p(y_{n-1}, y_n)$. *Then by* (2.5) *we have*

$$p(z_n, z_{n+1}) \leq tp(z_{n-2}, z_{n-1}).$$

Suppose

$$p(z_n, z_{n+1}) > p(y_{n-1}, y_n) \tag{2.9}$$

From (2.8) *and Definition 1.2 the following are the possibilities:*

$$p(z_n, y_n) \leq \frac{1}{2}p(y_{n-1}, y_n) \quad \text{or} \tag{2.10}$$

$$p(z_n, y_{n-1}) < \frac{1}{2}p(y_{n-1}, y_n). \tag{2.11}$$

Assume (2.10) *holds. Suppose, without loss of generality,* $y_n \in Fx_{n-1}$. *Then*

$$z_{n-1} = y_{n-1} = fx_{n-1} \in Gx_{n-2}, \quad y_n \neq z_n = fx_n.$$

By construction of $\{y_n\}$ *as in (iva), we have*

$$y_{n+1} = z_{n+1} = fx_{n+1} \in Gx_n$$

such that

$$p(y_n, y_{n+1}) \leq qH_p(Fx_{n-1}, Gx_n).$$

By (2.2) *and* (2.4) *we have*

$$\begin{aligned} p(y_n, y_{n+1}) &\leq qH_p(Fx_{n-1}, Gx_n) \\ &\leq t\max\{p(fx_{n-1}, fx_n), p(fx_{n-1}, Fx_{n-1}), p(fx_n, Gx_n), \\ &\quad \frac{p(fx_{n-1}, Gx_n) + p(fx_n, Fx_{n-1})}{2}\} \\ &\leq t\max\{p(fx_{n-1}, z_n), p(y_{n-1}, y_n), p(z_n, z_{n+1}), \\ &\quad \frac{p(y_{n-1}, z_{n+1}) + p(y_n, z_n)}{2}\} \end{aligned} \tag{2.12}$$

Since by (2.8) *and* (2.9), $p(y_{n-1}, z_n) < p(y_{n-1}, y_n) < p(z_n, z_{n+1})$, *we have*

$$p(y_{n-1}, z_{n+1}) + p(z_n, y_n) \leq p(y_{n-1}, z_n) + p(z_n, y_n) - p(z_n, z_n)$$

$$\begin{aligned} &+ p(z_n, z_{n+1}) \\ &= p(y_{n-1}, y_n) + p(z_n, z_{n+1}) < 2p(z_n, z_{n+1}). \end{aligned} \tag{2.13}$$

Thus, from (2.12) *we have*

$$p(y_n, y_{n+1}) \leq tp(z_n, z_{n+1}) \tag{2.14}$$

Since $y_{n+1} = z_{n+1}$, *from* (2.14), (2.10) *and* (2.7),

$$\begin{aligned} p(z_n, z_{n+1}) &\leq p(z_n, y_n) + p(y_n, y_{n+1}) \\ &\leq \frac{1}{2}p(y_{n-1}, y_n) + tp(z_n, z_{n+1}) \\ &\leq \frac{1}{2}tp(z_{n-2}, z_{n-1}) + \frac{2}{3}p(z_n, z_{n+1}) \end{aligned}$$

So we have

$$\begin{aligned} \frac{1}{3}p(z_n, z_{n+1}) &\leq \frac{1}{2}tp(z_{n-2}, z_{n-1}) \\ p(z_n, z_{n+1}) &\leq \frac{3}{2}tp(z_{n-2}, z_{n-1}). \end{aligned} \tag{2.15}$$

Similarly, (2.15) *can be established in case* $y_n \in Gx_{n-1}$.
We now consider a case when (2.11) *holds.*
Suppose, without loss of generality that $y_n \in Fx_{n-1}$. *Then*

$$y_{n-1} \in Gx_{n-2}.$$

By construction of $\{y_n\}$ *as in (ivb) and (2.4), we have*

$$\begin{aligned} p(y_{n-1}, y_{n+1}) &\leq qH_p(Gx_{n-2}, Fx_n) \\ &\leq t \max\left\{ \frac{3p(fx_{n-2}, fx_n)}{4}, p(fx_{n-2}, Gx_{n-2}), p(fx_n, Fx_n), \right. \\ &\quad \left. \frac{p(fx_{n-2}, Fx_n) + p(fx_n, Gx_{n-2})}{2} \right\} \\ &\leq t \max\left\{ \frac{3p(z_{n-2}, z_n)}{4}, p(z_{n-2}, z_{n-1}), p(z_n, z_{n+1}), \right. \\ &\quad \left. \frac{p(z_{n-2}, y_{n+1}) + p(z_n, y_{n-1})}{2} \right\} \end{aligned} \tag{2.16}$$

Since $y_{n-1} = z_{n-1}$ *from* (2.11) *and* (2.5),

$$p(z_n, y_{n-1}) < \frac{1}{2}p(y_{n-1}, y_n) \leq \frac{1}{2}tp(z_{n-2}, z_{n-1}) < \frac{1}{2} \times \frac{2}{3}p(z_{n-2}, z_{n-1}) = \frac{1}{3}p(z_{n-2}, z_{n-1}).$$

$$p(z_{n-2}, z_n) \leq p(z_{n-2}, z_{n-1}) + p(y_{n-1}, z_n) < \frac{4}{3}p(z_{n-2}, z_{n-1}).$$

$$p(z_{n-2}, y_{n+1}) \leq p(z_{n-2}, z_{n-1}) + p(y_{n-1}, y_{n+1}).$$

From (2.16), *we have*

$$p(y_{n-1}, y_{n+1}) \leq t \max\{p(z_{n-2}, z_{n-1}), p(z_n, z_{n+1}), \frac{2}{3}p(z_{n-1}, z_{n-1}) + \frac{1}{2}p(y_{n-1}, y_{n+1})\} \tag{2.17}$$

If we suppose that $p(y_{n-1}, y_{n+1}) > t \max\{p(z_{n-2}, z_{n-1}), p(z_n, z_{n+1})\}$, *then by* (2.17), *we have*

$$\begin{aligned} p(y_{n-1}, y_{n+1}) &\leq t[\frac{2}{3}p(z_{n-2}, z_{n-1}) + \frac{1}{2}p(y_{n-1}, y_{n+1})] \\ &\leq \frac{2}{3}tp(z_{n-2}, z_{n-1}) + \frac{1}{3}p(y_{n-1}, y_{n+1}) \end{aligned}$$

Thus, $p(y_{n-1}, y_{n+1}) \leq tp(z_{n-2}, z_{n-1})$ *which is in contradiction with* (2.9). *Therefore, we have*

$$p(y_{n-1}, y_{n+1}) \leq t \max\{p(z_{n-2}, z_{n-1}), p(z_n, z_{n+1})\}. \tag{2.18}$$

By the definition of partial metric (see Definition 1.2), (2.11), (2.5) *and* (2.18), *we have*

$$\begin{aligned} p(z_n, z_{n+1}) \leq\ & p(y_{n-1}, z_n) + p(y_{n-1}, y_{n+1}) \\ \leq\ & \frac{1}{2}p(y_{n-1}, y_n) + p(y_{n-1}, y_{n+1}) \\ \leq\ & \frac{1}{2}tp(z_{n-2}, z_{n-1}) + t \max\{p(z_{n-2}, z_{n-1}), p(z_n, z_{n+1})\} \\ \leq\ & \frac{1}{2}tp(z_{n-2}, z_{n-1}) + tp(z_{n-2}, z_{n-1}) + tp(z_n, z_{n+1}) \\ \leq\ & \frac{1}{2}tp(z_{n-2}, z_{n-1}) + tp(z_{n-2}, z_{n-1}) + \frac{2}{3}p(z_n z_{n+1}) \\ \leq\ & \frac{3}{2}tp(z_{n-2}, z_{n-1}) + \frac{2}{3}p(z_n, z_{n+1}) \end{aligned}$$

Therefore,

$$p(z_n, z_{n+1}) \leq \frac{3}{2}tp(z_{n-2}, z_{n-1}). \tag{2.19}$$

Similarly, if $y_n \in Gx_{n-1}$ *it can be shown that* (2.19) *holds.*

Thus, by (2.6), (2.7) and (2.19) for all $n \geq 2$ the following holds:

$$p(z_n, z_{n+1}) = \frac{3}{2}t \max\{p(z_{n-2}, z_{n-1}), p(z_{n-1}, z_n)\}.$$

Since $\frac{3}{2}t < 1$, by adapting the approach by Ćirić [7] we can show that $\{z_n\}$ is a Cauchy sequence, and therefore converges to a point s in K, as K is closed and therefore complete. Since $p(y_n, y_{n+1}) \leq p(z_n, z_{n+1})$, then $\{y_n\}$ converges to $s \in K$.

For each y_n denote by Y_n one of the subsets Fx_{n-1}, or Gx_{n-1} which contains y_n. Then by (2.5) and (2.18), we have

$$H_p(Y_n, Y_{n+1}) \leq hp(z_{n-1}, z_n),$$

or

$$H_p(Y_{n-1}, Y_{n+1}) \leq h \max p(z_{n-2}, z_{n-1}), p(z_{n-1}, z_n).$$

Since $\{z_n\}$ is a Cauchy sequence so is $\{Y_n\}$. Therefore, $\{Y_n\}$ converges to Y, for some Y in $CB^p(X)$, as $(CB^p(X), H_p)$ is a complete partial metric space. So we have

$$\begin{aligned} p(s, Y) \leq & p(s, z) \quad \text{for some} \quad z \in Y \\ & \leq p(s, y_n) + p(y_n, z) - p(y_n, y_n) \\ & \leq p(s, y_n) + H_p(Y_n, Y) - H_p(Y_n, Y_n) \\ & \leq p(s, s) \quad \text{as} \quad n \quad \text{approaches} \quad \infty. \end{aligned}$$

Therefore,

$$p(s, Y) = p(s, s) \Rightarrow s \in Y, \quad \text{since} \quad Y \quad \text{is closed.} \tag{2.20}$$

We now consider two subsequences $\{y_{n_j}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$, defined by $y_{n_j} \in Fx_{n_{j-1}} \cap K$ and $y_{n_k} \in Gx_{n_{k-1}} \cap K$. Clearly, by construction of $\{y_n\}$ at least one of the subsequences is infinite. Suppose, without loss of generality, that $\{y_{n_j}\}$ is infinite. For the sake of simplicity, we denote $\{y_{n_j}\}$, $\{x_{n_j}\}$, and $\{x_{n_{j-1}}\}$ by $\{y_j\}$, $\{x_j\}$, $\{x_{j-1}\}$, respectively.

ffx_j and Ffx_{j-1} are clearly well defined, as $fx_n = z_n \in K$. Set

$$L_j = p(fy_j, Ffx_{j-1})$$

and

$$R_j = H_p(Ffx_{n-j}, Fx_{j-1}).$$

Since $y_j \in Fx_{j-1}$, $y_{j+1} \in Gx_j$, then by (2.4) we have,

$$\begin{aligned} R_j \leq & \ H_p(Yj, Y_{j+1}) + H_p(Ffx_{j-1}, Gx_j) \\ \leq & \ H_p(Y_j, Y_{j+1}) + h \max \{p(ffx_{j-1}, fx_j), p(ffx_{j-1}, fy_j) + L_j, \\ & \ p(fx_j, Gx_j), p(ffx_{n-1}, Gx_j) + p(fx_j, Ffx_{j-1})\}. \end{aligned} \tag{2.21}$$

Hence as $fx_j = y_j \in Fx_{j-1}$, $y_{j+1} \in Gx_j$ and $L_j \leq p(fy_j, y_j) + R_j$, we have

$$\begin{aligned} Rj \leq & H_p(Y_j, Y_{j+1}) + \frac{2}{3} \max \{p(ffx_{j-1}, y_j), p(ffx_{j-1}, Ty_j) + p(fy_j, y_j) + R_j, \\ & p(y_j, y_{j+1}), p(ffx_{j-1}, y_{j+1}) + R_j\}. \end{aligned}$$

Therefore,

$$\begin{aligned} R_j \leq & 3H_p(Y_j, Y_{j+1}) + 2\left[p(ffx_{j-1}, y_j) + p(ffx_{j-1}, fx_j)\right. \\ & \left. + p(fy_j, y_j) + p(y_j, y_{j+1}) + p(ffx_{j-1}, y_{j+1})\right]. \end{aligned}$$

Since $Y_n \to Y$, $fx_j \to s$, $y_{j+1} \to s$, and as f is continuous, then the sequence $\{R_j\}$ is bounded and hence $\{\sup_j R_j\}$ is a convergent sequence. Also, since F and f are weakly compatible mappings (see Definition 1.16), and as $fx_{j-1} = z_{j-1} \in K$ and $\lim_{n\to\infty} p(y_j, fx_{j-1}) = 0$, we have

$$\limsup_{j\to\infty} L_j \leq \limsup_{j\to\infty} R_j, \tag{2.22}$$

and

$$\limsup_{j\to\infty} p(fy_j, fx_{j-1}) \leq \limsup_{j\to\infty} R_j. \tag{2.23}$$

We denote $\limsup_{j\to\infty} R_j$ by R. Taking the upper limit in (2.21) and using (2.22), (2.23) yields

$$R \leq \frac{2}{3}\max\{R, R, H_p(Y,Y), R\}.$$

So we have $R \leq \frac{2}{3}R$, since $H_p(Y,Y) \leq R$. Therefore, $R = 0$. Using (2.23), we have $p(fs, s) = 0$. This implies $s = fs$ (see Definition 1.2) (P3).
By (2.4), as $y_j \in Fx_{j-1}$ we have,

$$\begin{aligned} p(yj, Gs) &\leq H_p(Fx_{j-1}, Gs) \\ &\leq h\max\{p(fx_{n-1}, s), p(fx_{j-1}, Fx_{j-1}), p(s, Gs), p(fx_{j-1}, Gs) \\ &\quad + p(s, Fx_{j-1})\}. \end{aligned}$$

Taking the limit as $j \to \infty$ and using (2.20) gives,

$$p(s, Gs) \leq \frac{2}{3}\max\{p(s,s), p(s,s), p(s, Gs), p(s, Gs) + p(s,s)\}.$$

So we have $p(s, Gs) \leq \frac{2}{3}[p(s, Gs) + p(s,s)]$, since $p(s,s) \leq p(s, Gs)$.
Hence, $p(s, Gs) = 0 \Rightarrow p(s, Gs) = p(s,s)$, and therefore $s \in Gs$.
By (2.1) we have,

$$p(Fs, s) \leq H_p(Fs, Gs) \leq h\max\{p(s,s), p(s, Fs), p(s,s), p(s,s) + p(s, Fs)\}$$

and hence,

$$p(s, Fs) = 0 \Rightarrow p(s, Fs) = p(s,s) \Rightarrow s \in Fs.$$

Therefore, $s = fs \in Fs \cap Gs$, as desired.

Remark 2.3. If $FK \subseteq K$, $GK \subseteq K$ and $fK \subseteq K$, then Theorem 2.1 holds without the requirement of metrically convex property of complete partial metric space.

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