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Controlling Ultimate Ruin Probability by Quota-share Reinsurance Arrangements

Juma Kasozi[1], **Charles Wilson Mahera**[2] **and Fred Mayambala**[3]

[1,3]Department of Mathematics
Makerere University
P.O. Box 7062, Kampala, Uganda
kasozi@math.mak.ac.ug, fmayambala@math.mak.ac.ug

[2]Department of Mathematics
University of Dar es Salaam
P.O. Box 35062, Dar es Salaam, Tanzania
cwmahera@maths.udsm.ac.tz

ABSTRACT

A basic insurance model is perturbated by a diffusion. We take this model to represent the wealth dynamics of an insurance company. The model is compounded by another return on investments process of the Black-Scholes type. Both models form the risk process used in this work. Further, to manage her risk levels, the company enters into quota-share reinsurance arrangements with a reinsurer. We derive a second-order Volterra integro-differential equation which we transform into a linear Volterra integral equation of the second kind. We have solved the equations numerically using the block-by-block method for different retention levels for the chosen parameters. Results show that quota-share reinsurance improves the survival of the insurer.

Keywords: Ultimate Ruin probability, HJB equation, Volterra equations, Block-by-block method, Quota-share reinsurance.

2000 Mathematics Subject Classification: 91B30.

1 Introduction

If the individual risk is unmanageable, or if the expected loss levels in the entire insurance portfolio is too heavy, the cedant can transfer part of the risk to a second insurance carrier, the reinsurer thus offering additional underwriting capacity for cedants and also reducing the probability of the cedant's ruin. Quota-share type of proportional reinsurance is a treaty between the ceding company (insurer) and the reinsurer to share premiums and losses with the same proportion, thus, for some $b \in (0,1]$:

$$\begin{cases} \text{insured}'\text{s claim } S: \ bS \text{ paid by cedant}, \ \ (1-b)S \text{ paid by the reinsurer}, \\ \\ \text{premium } \tilde{P}: b\tilde{P} \text{ kept by the cedant}, \ \ (1-b)\tilde{P} \text{ remitted to the reinsurer}. \end{cases}$$

We make generalisations to the well known Cramér-Lundberg model (or the classical risk process or the surplus process) and also allow the company to invest into the financial world with returns of the Black and Scholes type. This paper focuses on ultimate ruin and one measure that insurance companies can take to control such a probability, that is, by signing up a proportional reinsurance treaty with a reinsurer. Other measures that could be used to achieve the same aims include investment of the capital, taking other forma of reinsurance, portfolio selection and volume control through the setting of premiums. This paper considers reinsurance coupled with investment as mechanisms for reducing the risk in an insurance portfolio. Reinsurance is essential not only from a legal standpoint but also due to the fact that the risk left to the first-line insurer is substantially reduced since the first-line insurer passes on some of its premium income to a reinsurer who, in turn, covers a certain proportion of the claims that occur. The time of ruin is the first time that the surplus of the cedant enters $(-\infty, 0)$ and the associated probability is referred to as the ultimate ruin probability. Ruin is a technical term which does not necessarily mean that the company is bankrupt but rather prompts the company to take action regarding its solvency status.

The problem of minimizing the ruin probability, when the insurance company has the possibility of investing part of its surplus into a market of stocks and bonds and of taking out proportional reinsurance for claims, has been extensively studied in different forms e.g. (Taksar and Markussen, 2003). It is rare to obtain an explicit expression for the ruin probability $\psi(y)$ in a jump-diffusion risk model set up, much of the literature focuses on the estimation of the ruin probability using numerical methods as well as upper bound estimation. In this paper, numerical methods will be used to compute ultimate ruin probabilities when the company goes into a quota-share reinsurance arrangement. In the literature, there exists no general rule asserting superiority of either quota-share-type or surplus-type reinsurance above the other. Other useful studies that allow for reinsurance and investments include (Irgens and Paulsen, 2004) and (Ma, Bai and Liu, 2008).

In this work, we take $b \in (0, 1]$ for reinsurance as the control variable in the presence of investments. Those dealing with reinsurance and investments all consider both the reinsurance and investment strategies as control variables. In other words, they determine the values of the retention level $b \in (0, 1]$ and the investment level $a \in [0, 1]$ (where a is the proportion of the insurer's surplus invested in a risky asset) that minimize the probability of ultimate ruin of the company).

The models in this work result into a Volterra integral equation (VIE) of the second kind and according to (Press, Teukolsky, Vetterling and Flannery, 1992), the block-by-block method, is the best of the higher order methods for solving a Volterra integral equation of the second kind. The block-by-block methods are essentially extrapolation procedures which produce a block of values at a time and have the advantage over linear multi-step and step-by-step methods in that they can be of high order and still be self-starting. Apart from not requiring special starting procedures, block-by-block methods are simple to use and switching step-size presents no problem (Linz, 1985). We take advantage of (Paulsen, Kasozi and Steigen, 2005) where they derived a linear Volterra integral equation of the second kind and applied an order-four block-

by-block method in conjunction with Simpson's rule to solve the Volterra integral equations of the second kind.

This paper is organized as follows. Section 2 contains the models to be studied. In Section 3, we give the HJB equation and verification theorems for the ruin probabilities under proportional re-insurance, as well as the corresponding Volterra integro-differential equation (VIDE) which we reduce to a linear Volterra integral equation (VIE) of the second kind by means of integration by parts. The main theorem is also stated and proved in this section. Section 4 contains the methods and numerical results, the last section contains concluding remarks and possible extensions.

2 The Models

All stochastic quantities and random variables are defined on a large enough stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$ satisfying the usual conditions, i.e. $\mathcal{F}_t$ is right continuous and $\mathbb{P}$-complete, $\mathbb{P}$ is the probability measure defined on $\mathcal{F}$. The notation followed in this paper is, in part, that of Paulsen (2008).

The surplus process, $\{P_t\}_{t\in[0,\infty)}$ is given by

$$P(t) = y + pt + \sigma_P W_P(t) - \sum_{i=1}^{N_P(t)} S_{P,i}, \quad t \geq 0; \tag{2.1}$$

where premiums are assumed to be collected over the time with constant rate p, W_P is a standard Wiener process independent of the compound Poisson process $\sum_{i=1}^{N_P(t)} S_{P,i}$, $\{S_{P,i}\}$ is the aggregate claim amount process and $\{N_P(t)\}$ is a homogeneous Poisson process (with intensity λ). The claim sizes are a sequence of strictly positive independent and identically distributed (iid) random variables $\{S_{P,i}\}_{i\in\mathbb{N}}$ which are independent of the claim arrival and claim number processes. We denote by F the distribution function of $S_{P,i}$, by $\mu = \mathbb{E}[S_{P,i}]$ its mean value and by $M_S(r) = \mathbb{E}\left[e^{rS_{P,i}}\right]$ its moment-generating function. We will assume that $F(0)$=0 and that at least one of σ_P or λ is non-zero.

The Brownian term $\sigma_P W_P$ represents random variations in the surplus process. It should be noted that, given an initial surplus y, when there is no volatility in the surplus and claim amounts (that is, when $\sigma_P = 0$), equation (2.1) becomes the classical risk process (or the Cramér-Lundberg model). Equation (2.1) is referred to as the classical risk process perturbed by a diffusion or a diffusion-perturbated classical risk process.

Now suppose that the insurer can enter a reinsurance treaty with a reinsurer. Reinsurance is of two types: *Random walk* type reinsurance which includes proportional, excess-of-loss (XL) and stop-loss reinsurance, and *extreme value* type reinsurance which includes largest claims and ECOMOR reinsurance (Excédent du coût moyen relatif or 'excess of the average cost'). While the literature generally points to non-proportional forms of reinsurance (such as stop-loss or XL reinsurance) as more efficient, this work focuses on proportional reinsurance which is the easiest way of covering an insurance portfolio.

Generally, let the stochastic process $\{b(t)\}_{t\in[0,\infty)}$, where $b(t)$ is the *retention level* at time t for reinsurance, represent a reinsurance strategy with values in $(0,1]$. This means that the insurer pays $b(t)S_{P,i}$ of the i-th claim $S_{P,i}$ occurring at time t $(i \in \mathbb{R}^+)$, while the reinsurer is liable for $(1-b(t))S_{P,i}$. Similarly, if p_i is the premium corresponding to the i-th policy, then the reinsurer receives $(1-b(t))p_i$ while the cedant retains $b(t)p_i$. For proportional reinsurance, however, the retention level, is constant over the entire insurance portfolio, that is, $b(t) = b$ for all risks occurring at any time t. Thus under a quota-share proportional reinsurance contract, the reinsurer is liable for $(1-b)S_{P,i}$ while the cedant covers $bS_{P,i}$ of the i-th claim occurring at time t. The factor $(1-b)$ is the *cession percentage* for the reinsurance which represents the proportion of claims ceded to the reinsurer as well as the share of premiums the reinsurer is to receive from the first-line insurer.

The model with proportional reinsurance is

$$P^b(t) = bpt + b\sigma_P W_P(t) - \sum_{i=1}^{N_P(t)} bS_{P,i} \tag{2.2}$$

having dynamics

$$dP^b(t) = bpdt + b\sigma_P \xi(t)dt - d\left(\sum_{i=1}^{N_P(t)} bS_{P,i}\right)$$

where $\xi(t)$ is a white noise process. If $b = 1$ then there is no reinsurance. If the expected value premium principle is used, then for the reinsurance a premium rate of $(1-b)(1+\theta)\lambda\mathbb{E}[S]$ has to be paid, where θ is the reinsurer's safety loading. This model was studied by (Ma et al., 2008) who obtained the minimal probability of ruin as well as the optimal proportional reinsurance strategy using the dynamic programming approach, as well as by (Schmidli, 2002) who allowed for investment into a risky asset and obtained, by means of an HJB equation, the optimal reinsurance and investment strategies for minimizing the ultimate ruin probability.

The model (2.2) is rather inadequate for modelling the real-world especially when there is a positive interest rate out there. One of the major limitations of (2.2) is that it does not account for interest earned on the reserve. Additionally, it does not account for long tail business with claims that are settled long after occurrence, nor does it include time-dependence or randomness of premium income and of the size of the portfolio. The insurer can invest part of its surplus, into say, a risk-free bond and a risky asset. Let the returns on investments process be:

$$R(t) = rt + \sigma_R W_R(t), \quad t \geq 0,\ R(0) = 0; \tag{2.3}$$

where r is the risk-free part of the investments so that $R(t) = rt$ implies that one unit invested now will be worth e^{rt} at time t; W_R is another Brownian motion so that the term $\sigma_R W_R(t)$ then takes account of random fluctuations in the investment returns; the $R(t) = rt + \sigma_R W_R(t)$ is the case in the famous Black and Scholes option pricing formula where the price of a stock is assumed to follow the stochastic differential equation

$$X_t = X_0 + \int_0^t X_s dR_s\,; \tag{2.4}$$

where X_0 is the stock price at $t = 0$. The process X is a geometric Brownian motion. The solution to ((2.4)) is the value of the stock at time t and is given by $X_t = X_0 \exp\{(r - \frac{1}{2}\sigma_R^2)t + \sigma_R W_{R,t}\}$.

The risk process is thus a careful combination of (2.2) and (2.3). It is the process $Y^b = \{Y^b(t)\}_{t\in[0,\infty)}$ which represents the insurance portfolio and has dynamics

$$dY^b(t) = dP^b(t) + Y^b(t^-)dR(t)$$

Mathematically, this means that Y^b is the solution of the linear SDE

$$Y^b(t) = y + P^b(t) + \int_0^t Y^b(s^-)dR(s) \tag{2.5}$$

where the non-negative constant $y = Y^b(0) > 0$ is the initial capital or surplus of the insurance company, $P^b(t)$ is the basic insurance (or surplus-generating) process in equation (2.2), $R(t)$ the investment process in equation (2.3) and $Y^b(t^-)$ denotes the insurer's surplus just prior to investing. Versions of this process have been extensively studied by several authors (see, (Paulsen et al., 2005) and (Paulsen, 2008).

(Paulsen, 1993), gave the solution to (2.5) as

$$Y^b(t) = \bar{R}(t)\left(y + \int_0^t \bar{R}^{-1}(s)dP^b(s)\right)$$

where

$$\bar{R}(t) = \exp\left\{\left(r - \tfrac{1}{2}\sigma_R^2\right)t + \sigma_R W_R(t)\right\}, \qquad t \geq 0$$

is the geometric Brownian motion so extensively used in Mathematical Finance and is the solution of $d\bar{R}(t) \overset{Itô}{=} r\bar{R}(t)dt + \sigma_R\bar{R}(t)dW_R(t)$, with $\bar{R}(0) = 1$.

3 HJB and Main Theorem

Let τ_b =inf$\{t > 0 : Y^b(t) < 0 | Y^b(0) = y\}$ be the time of ruin, with $\tau_b = \infty$ if $Y^b(t)$ remains positive, where $Y^b(t)$ is the risk process (equation (2.5) above) proportional reinsurance with strategy $b(t) = b \in (0, 1]$. The corresponding probability is the probability of ruin, thus $\psi^b(y) = \mathbb{P}\left(\tau_b < \infty | Y^b(0) = y\right) = 1 - \phi^b(y)$, here, $\phi^b(y)$ is the probability of survival.

The objective of this work is to find the minimal ruin probability $\psi(y) = \inf_{b\in(0,1]} \psi^b(y)$, and the optimal retention percentage b^* s.t. $\psi^{b^*}(y) = \psi(y)$, considered optimal if the percentage minimises the ruin probability. Since the investment generating process $R(t)$ follows (2.3), it follows that under weak assumptions the ruin probability $\psi(y)$ is twice continuously differentiable on $(0, \infty)$ and is a solution to the equation

$$\mathcal{A}\psi(y) = -\lambda_P \bar{F}(y) \tag{3.1}$$

(where $\bar{F}(y) = 1 - F(y)$) with boundary conditions $\lim_{y\to\infty} \psi(y) = 0$ and $\psi(y) = 0$ if $\sigma_P > 0$ (see Theorem 3.1 below). Here $\mathcal{A}$ is the integro-differential operator

$$\mathcal{A}g(y) = \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right) g''(y) + (ry + bp)g'(y) + \lambda_P \int_0^\infty (g(y - bs) - g(y))\, dF(s) \quad (3.2)$$

Sometimes it is more convenient to work with the survival probability $\phi(y) = 1 - \psi(y)$, in which case (3.1) becomes

$$\mathcal{A}\phi(y) = 0.$$

The integro-differential operator (3.2), which is the infinitesimal generator for Y^b, does not easily give rise to explicit analytical solutions and hence, the numerical methods. (Paulsen and Gjessing, 1997) proved that:

Theorem 3.1. *Let $\tau_b = inf\{t > 0 : Y^b(t) < 0 | Y^b(0) = y\}$ be the time of ruin where we set $\tau_b = \infty$ if $Y^b(t) \geq 0\ \forall\ t$. With the above notation, assume that $\psi(y)$ is bounded and twice continuously differentiable on $y \geq 0$ with a bounded first derivative there, where at $y = 0$ is meant the right-hand derivative. If $\psi(y)$ solves $\mathcal{A}\psi(y) = 0$ on $y > 0$, together with the boundary conditions*

$$\begin{aligned} \psi(y) &= 1\ on\ y < 0, \\ \psi(0) &= 1\ if\ \sigma_P^2 > 0, \\ \lim_{y\to\infty} \psi(y) &= 0, \end{aligned}$$

then

$$\psi(y) = \mathbb{P}(\tau_b < \infty)$$

Note: (Paulsen et al., 2005) used the probability of survival $\phi(y) = 1 - \psi(y)$ with boundary conditions

$$\begin{aligned} \phi(y) &= 0\ on\ y < 0, \\ \phi(0) &= 0\ if\ \sigma_P^2 > 0, \\ \lim_{y\to\infty} \phi(y) &= 1 \end{aligned} \quad (3.3)$$

Let $(0, h]$ be a small interval, and suppose that for each surplus $y(h) > 0$ at time h we have a reinsurance strategy b^ε s.t. $\phi^{b^\varepsilon}(y(h)) > \phi(y(h)) - \varepsilon$. We let $b(t) = b \in (0, 1]$ for $t \leq h$. Then

$$\begin{aligned} \phi(y) \geq \phi^b(y) &= \mathbb{E}\left[\phi^{b^\varepsilon}\left(Y^b(h)\right)\mathbf{1}_{\{\tau_b > h\}}\right] \\ &= \mathbb{E}\left[\phi^{b^\varepsilon}\left(Y^b(\tau_b \wedge h)\right)\right] \\ &\geq \mathbb{E}\left[\phi\left(Y^b(\tau_b \wedge h)\right)\right] - \varepsilon \end{aligned} \quad (3.4)$$

Because ε is arbitrary, we can choose $\varepsilon = 0$. By Itô's formula, provided that $\phi(y)$ is twice continuously differentiable, we have

$$\begin{aligned} \phi\left(Y^b(\tau_b \wedge h)\right) = \phi(y) &+ \int_0^{\tau_b \wedge h} \left\{(ry + bp)\phi'\left(Y^b(s)\right) + \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''\left(Y^b(s)\right)\right. \\ &+ \left.\lambda\left[\int_0^y \phi\left(Y^b(s) - a(s,b)\right) dF(s) - \phi\left(Y^b(s)\right)\right]\right\} ds \end{aligned} \quad (3.5)$$

where $a(s,b) = bs$ denotes the part of the claim $S_{P,i}$ paid by the cedent. Substituting (3.5) into the expected value (3.4) yields

$$\mathbb{E}\left[\int_0^{\tau_b \wedge h} \left\{(ry+bp)\phi'\left(Y^b(s)\right) + \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''\left(Y^b(s)\right) + \lambda\left[\int_0^y \phi\left(Y^b(s)-bs\right)dF(s) - \phi\left(Y^b(s)\right)\right]\right\} ds\right] \leq 0$$

Dividing by h and letting $h \to 0$ yields, provided that limit and expectation are interchangeable,

$$(ry+bp)\phi'(y) + \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''(y) + \lambda\left[\int_0^y \phi\left(y-bs\right)dF(s) - \phi(y)\right] \leq 0$$

which must hold $\forall\ b \in (0,1]$, that is,

$$\sup_{b\in(0,1]}\left\{(ry+bp)\phi'(y) + \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''(y) + \lambda\left[\int_0^y \phi\left(y-bs\right)dF(s) - \phi(y)\right]\right\} \leq 0$$

Suppose that there is an optimal strategy $b \in (0,1]$ s.t. $\lim\limits_{t\downarrow 0} b(t) = b(0)$. Then, as above,

$$\mathbb{E}\left[\int_0^{\tau_b \wedge h} \left\{(ry+bp)\phi'\left(Y^b(s)\right) + \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''\left(Y^b(s)\right) + \lambda\left[\int_0^y \phi\left(Y^b(s)-bs\right)dF(s) - \phi\left(Y^b(s)\right)\right]\right\} ds\right] = 0$$

Dividing by h and letting $h \to 0$ yields
$(ry+b_0p)\phi'(y) + \frac{1}{2}\left(\sigma_R^2 y^2 + b_0^2\sigma_P^2\right)\phi''(y) + \lambda\left[\int_0^y \phi(y-b_0 s)dF(s) - \phi(y)\right] = 0$ which motivates the HJB equation.

$$\sup_{b\in(0,1]}\left\{(ry+bp)\phi'(y) + \frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''(y) + \lambda\left[\int_0^y \phi\left(y-bs\right)dF(s) - \phi(y)\right]\right\} = 0 \quad (3.6)$$

with boundary conditions $\phi(y) = 0$ on $y < 0$ and $\lim\limits_{y\to\infty}\phi(y) = 1$ (see Theorem 3.1). The function $\phi(y)$ will satisfy (3.6) only if $\phi(y)$ is strictly increasing, strictly concave, twice continuously differentiable and satisfies $\phi(y) \to 1$ for $y \to \infty$. In the following, therefore, $\phi(y)$ will be assumed to be strictly increasing. This is consistent with the smoothness assumption and the intuition that the more wealth there is (through investment), the higher the probability that the insurer will survive. It will also be assumed that $\phi(y)$ is concave. To ensure smoothness and concavity, the claim density function must be locally-bounded. The following results provide verification of the existence and property of the solution of the HJB equation (3.6).

Proposition 3.2. ***(Existence of solution)*** *Let the claim-size distribution have a locally-bounded density. Then the HJB equation has a bounded twice continuously differentiable solution* $\phi \in C^2(0,\infty) \cap C^1[0,\infty)$.

The proof of this result has been given in (Hipp and Plum, 2000).

Proposition 3.3. ***(Property of the solution)*** *If* $\phi(y)$ *is twice continuously differentiable and solves the HJB equation* (3.6)*, then it is strictly increasing and strictly concave.*

This result has been proved in Hipp and Plum (2000).

The integro-differential equation for the survival probability $\phi(y)$, which follows from the HJB equation (3.6), is of the form $\mathcal{A}\phi(y) = 0$ (since, by (3.3), $\phi(y) = 0$ for $y < 0$), where $\mathcal{A}$ is the infinitesimal generator (3.2) of the underlying risk process, that is,

$$\frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''(y,b) + (ry + bp)\phi'(y,b) + \lambda_P \int_0^y \phi(y - bs, b)dF(s) - \lambda_P\phi(y) = 0, \quad (3.7)$$

for $0 < y \leq \infty$. Equation (3.7) could then be rewritten as

$$\frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi''(y) + (ry + bp)\phi'(y) + \lambda \int_0^y \phi(y - bs)dF(s) - \lambda\phi(y) = 0, \quad (3.8)$$

for $0 < y \leq \infty$. Equation (3.8) is a second-order integro-differential equation of the Volterra type (VIDE). However, repeated integration by parts transforms it into an ordinary Volterra integral equation (VIE) that will be used in this study.

Theorem 3.4. *The integro-differential equation* (3.8) *can be represented as the Volterra integral equation of the second kind*

$$\phi(y) + \int_0^y K(y,s)\phi(s)ds = \alpha(y) \quad (3.9)$$

where

$$\begin{aligned} K(y,s) &= 2\frac{(2r - 3\sigma_R^2 + \lambda)s + bp + \lambda G(\frac{y-s}{b}) - (r - \sigma_R^2 + \lambda)y}{\sigma_R^2 y^2 + b^2\sigma_P^2}, \\ \alpha(y) &= \frac{b^2\sigma_P^2\phi'(0)y}{\sigma_R^2 y^2 + b^2\sigma_P^2} \text{ for } \sigma_P^2 > 0. \end{aligned}$$

with $G(s) = \int_0^s F(v)dv$.

If $\sigma_P^2 = \sigma_R^2 = 0$, *then*

$$\begin{aligned} K(y,s) &= -\frac{r + \lambda\bar{F}(\frac{y-s}{b})}{ry + bp} \\ \alpha(y) &= \frac{bp\phi(0)}{ry + bp} \end{aligned}$$

Proof. The proof for $b = 1$ is given in (Paulsen et al., 2005). Integrating equation (3.8) by parts w.r.t. to y on $[0, z]$ gives

$$\begin{aligned} \frac{1}{2} &\left(\sigma_R^2 z^2 + b^2\sigma_P^2\right)\phi'(z) - \frac{1}{2}b^2\sigma_P^2\phi'(0) - \sigma_R^2 \int_0^z y\phi'(y)dy + (rz + bp))\phi(z) - bp\phi(0) \\ &-r\int_0^z \phi(y)dy + \lambda\int_0^z\int_0^y \phi(y - bs)dF(s)dy - \lambda\int_0^z \phi(y)dy = 0 \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}\left(\sigma_R^2 z^2 + b^2\sigma_P^2\right)\phi'(z) - \frac{1}{2}b^2\sigma_P^2\phi'(0) - \sigma_R^2\left[z\phi(z) - \int_0^z \phi(y)dy\right] + (rz + bp)\phi(z) \\ &-bp\phi(0) - r\int_0^z \phi(y)dy + \lambda\int_0^z\int_0^y \phi(y - bs)dF(s)dy - \lambda\int_0^z \phi(y)dy = 0, \end{aligned}$$

which on further simplification becomes

$$\frac{1}{2}\left(\sigma_R^2 z^2 + b^2\sigma_P^2\right)\phi'(z) - \frac{1}{2}b^2\sigma_P^2\phi'(0) + \left((r-\sigma_R^2)z + bp\right)\phi(z) - bp\phi(0)$$
$$-\left(r+\lambda-\sigma_R^2\right)\int_0^z \phi(y)dy + \lambda\int_0^z\int_0^y \phi(y-bs)f(s)dsdy = 0. \tag{3.10}$$

Let $v = y - bs$, then equation (3.10) becomes

$$\frac{1}{2}\left(\sigma_R^2 z^2 + b^2\sigma_P^2\right)\phi'(z) - \frac{1}{2}b^2\sigma_P^2\phi'(0) + \left((r-\sigma_R^2)z + bp\right)\phi(z) - bp\phi(0)$$
$$-\left(r+\lambda-\sigma_R^2\right)\int_0^z \phi(y)dy + \frac{\lambda}{b}\int_0^z\int_{bv}^z f(\frac{y-v}{b})dy\phi(v)dv = 0.$$

which on integration on $[0, y]$ wrt z gives (with z:=s)

$$\frac{1}{2}\left(\sigma_R^2 y^2 + b^2\sigma_P^2\right)\phi(y) - \frac{1}{2}b^2\sigma_P^2\phi(0) - (\frac{1}{2}b^2\sigma_P^2\phi'(0) + bp\phi(0))y$$
$$+\int_0^y [\left(2r - 3\sigma_R^2 + \lambda)s+\right) + bp + \lambda G(\frac{y-s}{b}) - (\lambda + r - \sigma_R^2)y]\phi(s)ds = 0.$$

which is in the form (3.9) with with $G(s) = \int_0^s F(v)dv$. □

The following result proved in (Linz, 1985) is useful:

Theorem 3.5. *The Volterra integral equation of the second kind* (3.9) *has a unique solution* $\phi \in C([c, d])$ *for any continuous* α.

4 Methods and Results

We now present numerical solutions of the survival probability $\phi(y)$ using a fixed grid $y = 0, h, 2h, \ldots$. It is assumed throughout that the assumptions of Theorem 3.1 hold. For this to happen, by Theorem 3.4, it is necessary that $r > \frac{1}{2}\sigma_R^2$ (if $\phi(y)$ is the survival probability and $p > 0$, $r > 0$ and $\lambda > 0$, then $\phi(y) > 0$ iff $r > \frac{1}{2}\sigma_R^2$, and in this case $\phi(\infty) = 1$. When $r \leq \frac{1}{2}\sigma_R^2$, $\phi(y) = 0\ \forall\ y$). The numerical solution of the general linear Volterra integral equation of the second kind

$$g(y) + \int_0^y K(y, s)g(s)ds = \alpha(y), \tag{4.1}$$

where the kernel $K(y, s)$ and the forcing function $\alpha(y)$ are known functions and $g(y)$ is the unknown function to be determined. Equation (4.1), is of the form

$$g_n + h\sum_{i=1}^{n} w_i K_{n,i} g_i = \alpha_n$$

where g_i is the numerical approximation to $g(ih)$, $K_{n,i} = K(nh, ih)$, $g_n = g(nh)$ and $\alpha_n = \alpha(nh)$. The w_i are the integration weights. Here, the block-by-block method will be used in conjunction with Simpson's rule of integration (which is known to have an error of order 4, see Remark 4.1) to obtain solutions in blocks of two values. For a detailed description of this method, see e.g. (Paulsen et al., 2005).

Definition 4.1. (Convergence) Let $g_0(h), g_1(h), \ldots$ denote the approximation obtained by a given method using step-size h. Then a method is said to be convergent iff

$$|g_i(h) - g(y_i)| \to 0, \ for\ i = 0, 1, 2, \ldots, N$$

as $h \to 0$, $N \to \infty$, s.t. $Nh = a$.

Definition 4.2. (Order of convergence) A method is said to be of order q if q is the largest number for which there exists a finite constant C s.t.

$$|g_i(h) - g(y_i)| \leq Ch^q, \quad i = 0, 1, 2, \ldots, \text{ for all } h > 0$$

We need to show that the method we use converges and also establish its order of convergence. We need the following lemma which forms the basis for the theorem that follows (Linz, 1985).

Lemma 4.1. *If* $|\xi_n| \leq A\sum_{i=0}^{n-1} |\xi_i| + B, \quad A > 0, B > 0$ *then* $|\xi_n| \leq B(1 + A)^n$

The proof follows immediately by induction. As a corollary we have that, if $A = hK$ and $y = nh$, then

$$|\xi_n| \leq B\mathrm{e}^{Ky}$$

Remark 4.1. By Theorem 3.1 in (Paulsen et al., 2005) and from results in Chapter 7 of (Linz, 1985), it follows that for a fixed y so that $nh = y$, the solution satisfies

$$|g_n - g(y)| = O(h^4),$$

provided that g is four times continuously differentiable as is the case here by Theorem 2.4 in (Paulsen et al., 2005).

We now present some results obtained using the above methods for the exponential distribution. To improve on the computing time, programming language adopted is FORTRAN, and the DOUBLE PRECISION feature has been invoked to get high accuracy. Theorem (3.4) is about survival probability but the programs have been adjusted to output ultimate ruin probabilities since $\psi(y) = 1 - \phi(y)$. The method can be initiated using any value of $\phi(0)$. The values stablise at $g(\infty)$ that is used to scale the probabilities. The corresponding upper bound $\bar{y}$ must be large enough so that $\phi(\bar{y} - 999h)$ is virtually equal to 1. The step size used is $h = 0.01$ in all cases. The graphs have been constructed using MATLAB 6.5.

4.1 Quota-share Re-insurance in the basic model

Exp(μ) refers to the exponential density $f(s) = \mu\mathrm{e}^{-\mu s}$. Thus the distribution function for the exponential distribution is $F(s) = 1 - \mathrm{e}^{-\mu s}$ and $\bar{F}(s) = 1 - F(s) = \mathrm{e}^{-\mu s}$. The mean function for the Exp(μ) distribution is $\frac{1}{\mu}$. From Theorem (3.4), the kernel and forcing functions are respectively:

$$\begin{aligned} K(y, s) &= -\frac{\lambda\bar{F}(\frac{y-s}{b})}{bp} \\ \alpha(y) &= \phi(0) \end{aligned}$$

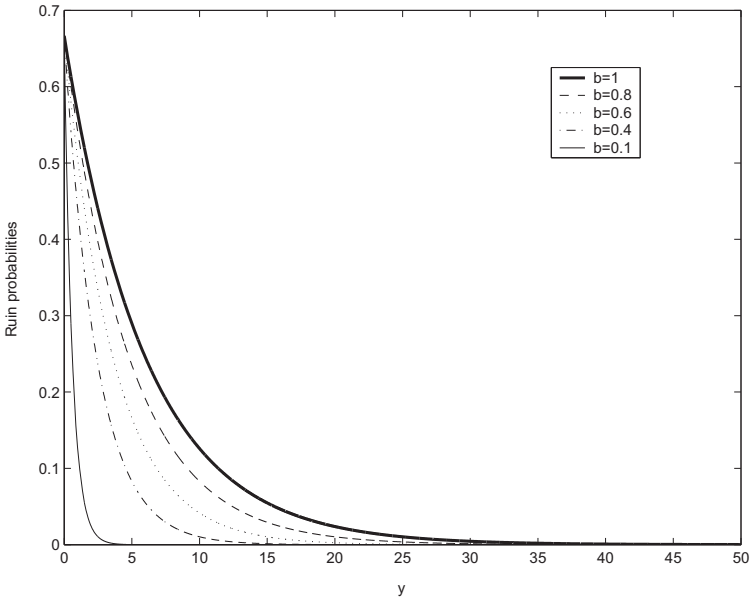


Figure 1: Ultimate Ruin Probabilities at different ceding levels for Exp(0.5) claims, $\lambda = 2, p = 6$.

The case $b = 1$ corresponds to no reinsurance and the results are same as those in (Paulsen et al., 2005). The subsequent ultimate ruin probabilities correspond to different re-insurance levels. It can be observed from Figure 1 that re-insurance reduces the probabilty of ruin or correspondingly, increases the probability of survival. This is the ultimate goal of re-insurance as precisely expounded in Section 1.

4.2 Quota-share Re-insurance under a constant force of interest

If the company invests part of the portfolio into a risk free asset so that one unit now returns e^{rt} at time t, then the kernel and forcing function change accordingly thus, from Theorem (3.4)

$$K(y,s) = -\frac{r+\lambda\bar{F}(\frac{y-s}{b})}{ry+bp}$$

$$\alpha(y) = \frac{bp\phi(0)}{ry+bp}$$

Figure 2 compared to the case without investment into a risk free asset in Figure 1 shows a reduction in the ruin probability as expected since the safe investment results into considerable returns into the company and hence, increased survival of the company. Evidently, $\psi(0) = 0.66666667$ for the case without the safe investment and reduces to $\psi(0) = 0.63979769$ if the company invests into a riskless asset. This trend goes for all other values of y.

If the company expects heavy claims, then we need to choose an appropriate distribution for

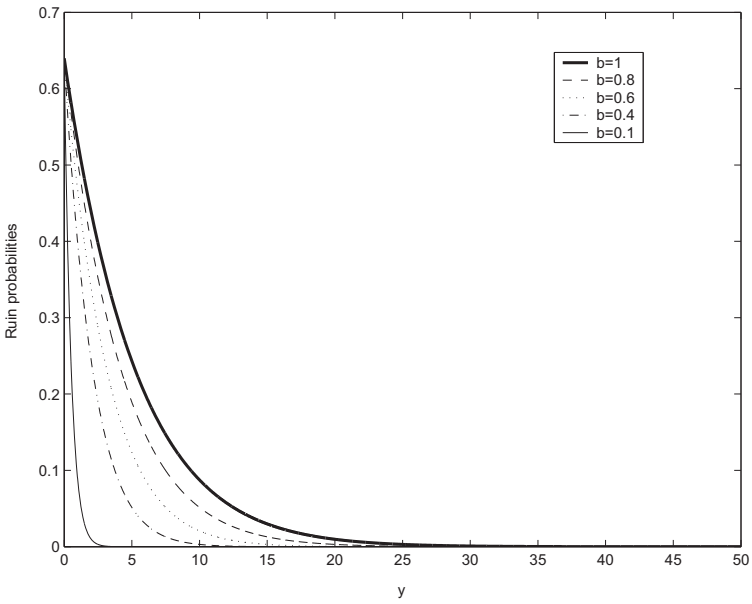


Figure 2: Ultimate Ruin Probabilities at different ceding levels for Exp(0.5) claims, $r = 5\%$, $\lambda = 2$, $p = 6$.

F. We have included results for heavy claims for only this case. The results in Figure 3 are for the Pareto distribution where

$$f(s) = (\rho - 1)^{\rho}\, \rho\, (\rho - 1 + s)^{-(1+\rho)}, \;\; s > 0, \; \rho > 1$$

and

$$\bar{F} = \left(\frac{\rho - 1}{\rho - 1 + s}\right)^{\rho}.$$

Figure 3 for Pareto distribution has similar explanation as the one for Exponential distribution. Still quota-share reinsurance improves the survival of the company.

4.3 Quota-share Re-insurance with investments of Black-Scholes type

Suppose there are fluctuations in the insurance model and the company undertakes investments of the Black-Scholes. Then, the kernel and forcing function can be deduced from Theorem (3.4) thus, for $\sigma_P^2 > 0$

$$K(y,s) = 2\frac{(2r - 3\sigma_R^2 + \lambda)s + bp + \lambda G(\frac{y-s}{b}) - (r - \sigma_R^2 + \lambda)y}{\sigma_R^2 y^2 + b^2\sigma_P^2},$$

$$\alpha(y) = \frac{b^2\sigma_P^2\phi'(0)y}{\sigma_R^2 y^2 + b^2\sigma_P^2}$$

with $G(s) = \int_0^s F(v)dv$. The case $\sigma_P^2 = 0$ has not been studied in this paper. It needs special handling. There is a singularity at 0 since $\alpha(y) = \frac{2bp}{\sigma_R^2 y}\phi(0)$. Here $\bar{y} = 1000$. Figure 4 shows the

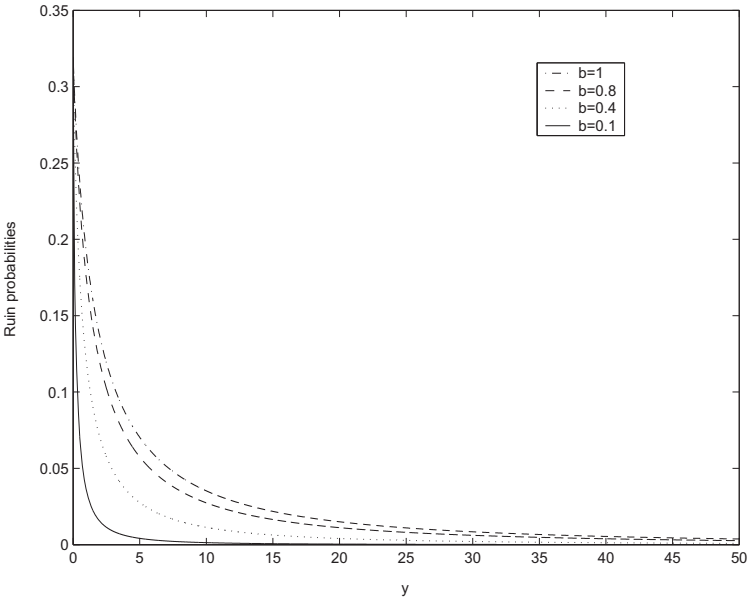


Figure 3: Ultimate Ruin Probabilities at different ceding levels for process parameters $\rho = 2$, $r = 5\%$, $\lambda = 2$, $p = 6$.

ruin probabilities for this case.

The data used to plot Figure 4 reveals lots of information. Without any reinsurance, that is, when $b = 1$, $\psi(5) = 0.51153634$. When $b = 0.8$, $\psi(5) = 0.40578113$. As b decreases, the ruin probability also decreases, as expected. For this case, it was found out that the value of $\psi(5)$ for $b = 0.1$ was higher than the $\psi(5)$ when $b = 0.2$. This means that the optimal value of b lies in the interval $(0.1, 0.2)$. The curves in Figure 4 depicts the same thing. Indeed, the curve for $b = 0.1$ is higher than that of $b = 0.2$ and $b = 0.4$.

5 Conclusions

An increase in the rate of return on investments, results into an increase in the survival probability or a reduction in the ruin probability. This is the same with an increase in the premium rate. The results are the same trend for both the small and large claim cases, as shown in the case of a constant force of interest. It can therefore be concluded that investments play a major role in helping insurance companies to minimize their probabilities of ultimate ruin. The more of their surplus insurance companies invest in risky and risk-free assets, the lower their ruin probabilities. Insurers can never reach a point (in terms of investment rate) beyond which it is not profitable to invest. However, as stock prices become more volatile (that is, as σ_R increases), the ruin probability also increases. Volatility is actually a measure of the riskiness

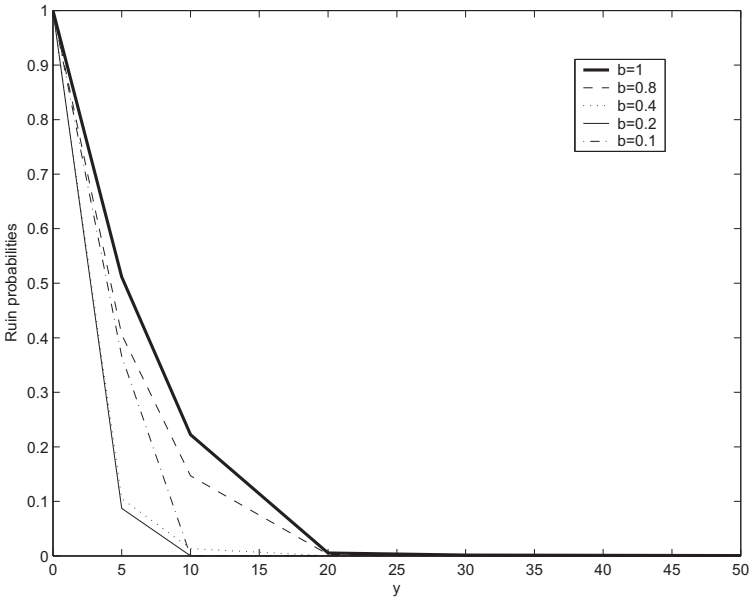


Figure 4: Ultimate Ruin Probabilities at different ceding levels for Exp(0.5) claims, $r = 30\%$, $\lambda = 1$, $p = 6$, $\sigma_P = \sigma_R = 0.001$.

of a stock. If the volatility of the stock price increases but the expected rate of return of the stock stays the same, then the insurer will find the reward for accepting the risk unattractive and would rather invest less in stocks and invest more in bonds. Conversely, if the volatility of the stock price decreases, then the insurer will receive the same return but with a lower risk and will find that it makes economic sense to invest in the stock. The results from the previous section indicate that quota-share (proportional) reinsurance does have a positive impact on the survival of insurance companies as it minimizes their ultimate ruin probabilities. The work can be extended to include jumps into the investment process and also using other forms of re-insurance arrangements.

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