Jñānābha, Vol. 48(2) (2018), 14-18

FIXED POINTS FOR HYBRID MAPPINGS SATISFYING AN IMPLICIT RELATION IN PARTIAL METRIC SPACES

By
Santosh Kumar
Department of Mathematics, College of Natural and Apllied Sciences
University of Dar es Salaam, Tanzania
`Email:drsengar2002@gmail.com`
Terentius Rugumisa
Faculty of Science, Technology and Environmental Studies
The Open University of Tanzania, Tanzania
`Email:rterentius@gmail.com`

(*Received : September* 23, 2018; *Revised: November* 17, 2018)

Abstract

In this paper, we prove a fixed point theorem for hybrid mappings in partial metric spaces. The theorem contains an altering distance function and involves an implicit relation satisfying the (E.A) - property. In doing so, we generalize a theorem by Popa and Patriciu.

2010 Mathematics Subject Classifications: 47H10, 54H25.
Keywords and phrases: Partial metric space, hybrid mappings, implicit relation, Hausdorff partial metric, (E.A) property.

1 Introduction and Preliminaries

The study of fixed point theorems involving implicit relations was initiated by Popa [5]. These types of theorems cater for a large class of fixed point theorems on mappings that obey a given set of conditions. Popa and Patriciu [6] proved a fixed point theorem for metric spaces for a pair of hybrid mappings involving altering distance and satisfying an implicit relation. In this study, we modify this theorem so that it applies for partial metric spaces.

The following preliminaries will be useful in the course of our work.

The partial metric, which is a generalization of the metric with the zero-self distance axiom relaxed is defined here.

Definition 1.1. [4] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$,*
(P1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(P2) $p(x,x) \leq p(x,y)$,
(P3) $p(x,y) = p(y,x)$,
(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Let $CB^p(X)$ be a family of all non-empty, closed and bounded subsets of a partial metric space (X,p), induced by the partial metric p. The set A is said to be a bounded subset in (X,p) if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$.

The Hausdorff partial metric is defined for multivalued mappings in partial metric spaces. We make use of the following definitions given by Aydi *et al.* [2].

Definition 1.2. [2] *For all* $A, B \in CB^p(X)$ *and* $x \in X$*, we define*
(i) $\quad p(x, A) = \inf\big\{p(x, a), a \in A\big\}$,
(ii) $\quad \delta_p(A, B) = \sup\big\{p(a, B) : a \in A\big\}$,
(iii) $\quad \delta_p(B, A) = \sup\big\{p(b, A) : b \in B\big\}$,
(iv) $\quad H_p(A, B) = \max\big\{\delta_p(A, B), \delta_p(B, A)\big\}$.
The mapping $H_p : CB^p \times CB^p \to [0, +\infty)$ *is called the partial Hausdorff metric.*

We now state some properties of mappings δ_p and H_p.

Lemma 1.1. [2] *Let* (X, p) *be a partial metric space. For any* $A, B \in CB^p(X)$ *we have*
(i) $\quad \delta_p(A, A) = \sup\{p(a, a) : a \in A\}$,
(ii) $\quad \delta_p(A, A) \leq \delta_p(A, B)$,
(iii) $\quad \delta_p(A, B) = 0$ *implies that* $A \subseteq B$,
(iv) *If* $b \in B$*, then* $p(a, B) \leq p(a, b) \leq \delta(a, B)$
(h1) $H_p(A, A) \leq H_p(A, B)$,
(h2) $H_p(A, B) = H_p(B, A)$,
(h3) $H_p(A, B) = 0$ *implies* $A = B$.

Let f and T be mappings with $f : (X, p) \to (X, p)$ and $T : (X, p) \to CB^p(X)$. A point $x \in X$ is said to be a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $C(f, T)$.

Inspired by Aamri and Moutawakil [1] and Kamran [3] we define the (E.A)-property in the context of hybrid mappings in partial metric spaces.

Definition 1.3. *Let* (X, p) *be a partial metric space. The mappings* $f : X \to X$ *and* $T : X \to CB^p(X)$ *are said to satisfy the (E.A)-property if there exists a sequence* $\{x_n\}$ *in* X *such that* $\lim fx_n = t \in A = \lim Tx_n$ *and* $p(t, t) = 0$.

Definition 1.4. *An altering distance is a mapping* $\varphi : [0, \infty) \to [0, \infty)$ *which satifies:*

(φ_1)*:* $\varphi(t)$ *is increasing and continuous,*
(φ_2)*:* $\varphi(t) = 0$ *if and only if* $t = 0$.

The function space $\mathfrak{F}_a$ is defined as follows in Popa and Patriciu [6].

Let $\mathfrak{F}_a$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ satisfying the following
(F1) : F is nondecreasing in variable t_1,
(F2) : $F(t, 0, 0, t, t, 0) \leq 0$ implies $t = 0$.

Examples of functions in $\mathfrak{F}_a$ are hereby stated.

Example 1.1. [6] $F(t_1, ..., t_6) = t_1 - \max\{t_2, (t_3 + t_4)/2, (t_5 + t_6)/2\}$.
(F1)*: Obviously.*
(F2) *:* $F(t, 0, 0, t, t, 0) = t/2 \leq 0$ *implies* $t = 0$.

Example 1.2. [6] $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$*, where* $a, b, c \geq 0$ *and* $b + c < 1$.
(F1)*: Obviously.*
(F2) *:* $F(t, 0, 0, t, t, 0) = t(1 - b) \leq 0$ *implies* $t = 0$.

Popa and Patriciu [6] proved the following theorem:

Theorem 1.1. *Let $f : (X, d) \to (X, d)$ and $T : (X, d) \to CB(X)$ be such that*

$$F\big(\phi(H(Tx, Ty)), \phi(d(fx, fy)), \phi(d(fx, Tx)), \phi(d(fy, Ty)), \phi(d(fx, Ty)), \phi(d(fy, Tx))\big) \leq 0 \tag{1.1}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_a$ and $\phi(t)$ is an altering distance. If $f(X)$ is a closed subset of X and (f, T) satisfy $(E.A) -$ property, then $C(f, T) \neq 0$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

In this study we modify Theorem 1.1 so that it applies to partial metric spaces.

2 Main Results

Now, we will extend Theorem 1.1 to partial metric spaces as follows:

Theorem 2.1. *Let (X, p) be a complete partial metric space. Let $f : (X, p) \to (X, p)$ and $T : (X, p) \to CB^p(X)$ be continuous mappings such that*

$$F(\varphi(H_p(Tx, Ty)), \varphi(p(fx, fy)), \varphi(p(fx, Tx)), \varphi(p(fy, Ty)), \varphi(p(fx, Ty)), \varphi(p(fy, Tx))) \leq 0 \tag{2.1}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_a$ and $\varphi(t)$ is an altering distance. If $f(X)$ is a closed subset of X and (f, T) satisfy $(E.A) -$ property, then $C(f, T) \neq \emptyset$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

Proof. From the Definition 1.3, there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = t \in A = \lim Tx_n$, with $p(t, t) = 0$. Since f, T are continuous mappings, we have $t = fx \in Tx$ where $x = lim_{n\to\infty} x_n$. This makes $x \in C(f, T)$, implying $C(f, T) \neq \emptyset$.

We have,

$$t = fx = ffx = ft \in Tx.$$

By (2.1) we have

$$\begin{aligned} &F(\varphi\big(H_p(Tx, Tt)), \varphi(p(fx, ft)), \varphi(p(fx, Tx)), \varphi(p(ft, Tt)), \\ &\qquad \varphi(p(fx, Tt)), \varphi(p(ft, Tx))\big) \leq 0 \\ \Rightarrow\ &F\big(\varphi(H_p(Tx, Tt)), \varphi(p(t, t)), \varphi(p(t, Tx)), \varphi(p(t, Tt)), \\ &\qquad \varphi(p(t, Tt)), \varphi(p(t, Tx))\big) \leq 0. \\ \Rightarrow\ &F\big(\varphi(H_p(Tx, Tt)), 0, 0, \varphi(p(t, Tt)), \varphi(p(t, Tt)), 0\big) \leq 0. \end{aligned} \tag{2.2}$$

As $t \in Tx$ we have $p(t, Tt) \leq H_p(Tx, Tt)$. Since F is non-decreasing in the first variable, (2.2) becomes

$$F\big(\varphi(p(t, Tt)), 0, 0, \varphi(p(t, Tt)), \varphi(p(t, Tt)), 0\big) \leq 0. \tag{2.3}$$

Using the (F_2) property we get

$$p(t, Tt) = 0 \ \Rightarrow t = ft \in Tt,$$

making $\lim fx_n = t$ a common fixed point of f and T. □

If $\varphi(t) = t$ we get the following corollary.

Corollary 2.1. *Let* (X,p) *be a complete partial metric space. Let* $f:(X,p)\to(X,p)$ *and* $T:(X,p)\to CB^p(X)$ *be continuous mappings such that*

$$\begin{aligned} F(H_p(Tx,Ty), p(fx,fy), p(fx,Tx), p(fy,Ty),& \\ p(fx,Ty), p(fy,Tx)) \le 0& \end{aligned} \tag{2.4}$$

for all $x,y\in X$, *where* $F\in\mathfrak{F}_a$. *If* $f(X)$ *is a closed subset of* X *and* (f,T) *satisfy* $(E.A)-$*property, then* $C(f,T)\neq\emptyset$. *Moreover, if* $fv=ffv$ *for* $v\in C(f,T)$, *then* f *and* T *have a common fixed point.*

We state an example for the use of this theorem.

Example 2.1. *Consider the implicit function* F *defined in Example* 1.1. *Let* $\varphi(t)=t$. *This leads to the following theorem.*

Theorem 2.2. *Let* (X,p) *be a complete partial metric space. Let* $f:(X,p)\to(X,p)$ *and* $T:(X,p)\to CB^p(X)$ *be continuous mappings such that*

$$\begin{aligned} H_p(Tx,Ty)\le\max\big\{p(fx,fy),&(p(fx,Tx)+p(fy,Ty))/2, \\ &(p(fx,Ty)+p(fy,Tx))/2\big\}. \end{aligned} \tag{2.5}$$

for all $x,y\in X$. *If* $f(X)$ *is a closed subset of* X *and* (f,T) *satisfy* $(E.A)-$ *property, then* $C(f,T)\neq\emptyset$. *Moreover, if* $fv=ffv$ *for* $v\in C(f,T)$, *then* f *and* T *have a common fixed point.*

Proof. In view of the $(E.A)-$ property on mappings f and T described in Definition 1.3, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} fx_n = t\in A = \lim_{n\to+\infty} Tx_n \text{ and } p(t,t)=0. \tag{2.6}$$

As f and T are continuous mappings, we have $fx=t\in Tx$ where $x=\lim_{n\to+\infty}x_n$. This makes x a coincidence point of f and T. Hence $C(f,T)\neq\emptyset$.

As x is a coincidence point, from the assumption we have

$$t=fx=ffx=ft\in Tx, \tag{2.7}$$

making t a fixed point of f.

To complete the proof, we need to show that t is also a fixed point of T, that is $t\in Tt$. We consider the following:

$$\begin{aligned} &H_p(Tx,Tt) \\ &\quad\le\max\left\{p(fx,ft),\frac{1}{2}\big(p(fx,Tx)+p(ft,Tt)\big),\frac{1}{2}\big(p(fx,Tt)+p(ft,Tx)\big)\right\} \\ &\quad=\max\left\{p(t,t),\frac{1}{2}\big(p(t,Tx)+p(t,Tt)\big),\frac{1}{2}\big(p(t,Tt)+p(t,Tx)\big)\right\} \\ &\quad=\frac{1}{2}\big(p(t,Tx)+p(t,Tt)\big). \end{aligned} \tag{2.8}$$

From (2.6), we have $p(t,t)=0$. In addition, from (2.7) we have $t\in Tx$ which implies $p(t,Tx)=0$. Thus (2.8) becomes

$$H_p(Tx,Tt)\le\frac{1}{2}p(t,Tt). \tag{2.9}$$

As $t \in Tx$, we have

$$\begin{aligned} p(t, Tt) &\leq H_p(Tx, Tt) \leq \frac{1}{2} p(t, Tt) \\ &\Rightarrow p(t, Tt) = 0 \\ &\Rightarrow t \in Tt. \end{aligned}$$

Hence t is also a fixed point of T. We have shown that t is a common fixed point of f and T. We have also shown that $p(t,t) = 0$. ☐

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflicts of interest.

References

[1] A. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.,* **270(1)** (2002),181-188.

[2] H. Aydi, M. Abbas and Vetro, C., Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology and its Applications,* **159(14)** (2012), 3234-3242.

[3] H. Kamran, Coincidence, fixed points for hybrid contractions, *J. Math. Anal. Appl.,* **229** (2004), 235-241.

[4] S. G. Matthews, , Partial metric topology, *Ann. N.Y. Acad. Sci.*, **728** (1994), 183-197.

[5] V. Popa, Some fixed point theorems for compatible mappings satisfying an implicit relation, *Demonstratio Math.,* **32** (1999), 157-163.

[6] V. Popa and A-M. Patriciu, Coincidence and Common Fixed Points for Hybrid Mappings Satisfying an Implicit Relation and Applications, *Thai Journal of Mathematics,* **15(1)** (2017), 81-89.