

Analysis and Dynamically Consistent Numerical Schemes for the SIS Model and Related Reaction Diffusion Equation

J. M.S. Lubuma, E. Mureithi, and Y. A. Terefe

Citation: AIP Conf. Proc. **1404**, 168 (2011); doi: 10.1063/1.3659917
View online: http://dx.doi.org/10.1063/1.3659917
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=1404&Issue=1
Published by the American Institute of Physics.

Related Articles

A study of similarity solutions for laminar swirling axisymmetric flows with both buoyancy and initial momentum flux
Phys. Fluids 23, 113601 (2011)

Combination synchronization of three classic chaotic systems using active backstepping design
Chaos 21, 043114 (2011)

Turbulent boundary layer flow subject to streamwise oscillation of spanwise wall-velocity
Phys. Fluids 23, 081703 (2011)

Low-dimensional model of turbulent Rayleigh-Bénard convection in a Cartesian cell with square domain
Phys. Fluids 23, 077101 (2011)

Microbubbles as contrast agent for in-line x-ray phase-contrast imaging
Appl. Phys. Lett. 99, 011101 (2011)

Additional information on AIP Conf. Proc.

Journal Homepage: http://proceedings.aip.org/
Journal Information: http://proceedings.aip.org/about/about_the_proceedings
Top downloads: http://proceedings.aip.org/dbt/most_downloaded.jsp?KEY=APCPCS
Information for Authors: http://proceedings.aip.org/authors/information_for_authors

ADVERTISEMENT



Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

Analysis and Dynamically Consistent Numerical Schemes for the SIS Model and Related Reaction Diffusion Equation

J. M-S Lubuma[1], E. Mureithi and Y. A. Terefe

Department of Mathematics and Applied Mathematics, University of Pretoria, South Africa

Abstract. The classical SIS epidemiological model is extended in two directions: (a) The number of adequate contacts per infective in unit time is assumed to be a function of the total population in such a way that this number grows less rapidly as the total population increases; (b) A diffusion term is added to the SIS model and this leads to a reaction diffusion equation, which governs the spatial spread of the disease. With the parameter $\mathscr{R}_0$ representing the basic reproduction number, it is shown that $\mathscr{R}_0 = 1$ is a forward bifurcation for the model (a), with the disease-free equilibrium being globally asymptotic stable when $\mathscr{R}_0$ is less than 1. In the case when $\mathscr{R}_0$ is greater than 1, traveling wave solutions are found for the model (b). Nonstandard finite difference (NSFD) schemes that replicate the dynamics of the continuous models are presented. In particular, for the model (a), a nonstandard version of the Runge-Kutta method having high order of convergence is investigated. Numerical experiments that support the theory are provided.

Keywords: Epidemiological models, local/global stability, nonstandard finite difference scheme, reaction diffusion equation
PACS: 87.10.Ed, 87.23cc

INTRODUCTION

It is well known that the mass action principle or incidence, though abundantly used in the modeling of infectious diseases, is not suitable since it relies on the unrealistic assumption that the contact rate of infection is proportional to the size of the total population. On the contrary, as discussed in [1], the standard incidence formulation is relevant. In this work, we consider general contact rates that capture well the tendency for the number of contacts made by an average infective in unit time to saturate [2]. To simplify the presentation, we restrict our study to the SIS model.

The basic reproduction number being expressed in terms of the general contact rate, we establish the main result that describes the dynamics of the SIS model. We design a nonstandard finite difference (NSFD) scheme that is dynamically consistent with the basic properties of the continuous model. Furthermore, we introduce a nonstandard version of the Runge-Kutta method of order 4, a result which to the best knowledge of the authors has not been investigated. In a second step, we consider a reaction diffusion system for the spatial spread of disease, the underlying space independent of the system being the SIS model with general contact rate. The well-posedness and the main dynamics of the diffusion system are investigated along the lines of [4, 5], while innovative NSFD schemes that replicate the dynamics are designed in accordance with [6, 7, 9]. In the concluding remark, we mention among other things our plan to extend this study to Volterra integral equation, as initiated in [10].

THE SIS MODEL WITH GENERAL CONTACT RATE

The SIS model for the spread of disease is well-known. The total population N is divided into two disjoint classes: individuals S susceptible to infection and infective individuals I, which both die naturally at the rate $\mu > 0$, while the disease induces death at the rate $\alpha \geq 0$ and susceptible individuals are recruited at the rate μK, where $K > 0$ is the carrying capacity. Susceptible individuals become infective after contact with infective individuals. Infective individuals return to the susceptible class at the rate $\delta > 0$ after an infective period. This study is specific in that the adequate contacts per infective in unit time grow less rapidly as the population increases. More precisely, following [2, 3], we assume that the contact rate C is a function of the population, $C \equiv C(N)$, which satisfies the following

[1] Corresponding author: jean.lubuma@up.ac.za

Application of Mathematics in Technical and Natural Sciences
AIP Conf. Proc. 1404, 168-175 (2011); doi: 10.1063/1.3659917
© 2011 American Institute of Physics 978-0-7354-0976-7/$30.00

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

conditions:

$$C(N) \text{ is bounded, } C'(N) \geq 0, \ \beta(N) := \frac{C(N)}{N} \text{ and } \beta'(N) \leq 0. \tag{1}$$

The SIS model with general contact rate reads then as:

$$S' = \mu K - \beta(N)IS - \mu S + \delta I \tag{2}$$
$$I' = \beta(N)IS - (\mu + \alpha + \delta)I. \tag{3}$$

The main dynamics of the SIS model are summarized in the following result:

Theorem 1 *1. The SIS model is a dynamical system on*

$$\Omega = \{(S,I) \in \mathbb{R}^2_+ : 0 \leq S + I = N \leq K\}.$$

2. *The basic reproduction number is* $\mathscr{R}_0 = \frac{C(K)}{\alpha+\mu+\delta}$.
3. *If* $\mathscr{R}_0 < 1$, *the disease-free equilibrium* $E_0 = (K,0)$ *is globally asymptotically stable- GAS.*
4. *If* $\mathscr{R}_0 > 1$, E_0 *is unstable and there exists a unique endemic equilibrium,* $E_\infty = (S_\infty, I_\infty)$, *which is locally asymptotically stable- LAS.*
5. *The following conservation law holds:*

$$\mu K - (\mu + \alpha)N \leq N' \leq \mu(K - N) \tag{4}$$

The relevance of the basic reproduction number in the statements in the theorem is intuitively seen if the model (2)–(3) is written in the equivalent form:

$$N' = \mu(K - N) - \alpha I \tag{5}$$
$$I' = \beta(N)\left[N\left(1 - \frac{1}{\mathscr{R}_0}\frac{C(K)}{C(N)}\right) - I\right]I. \tag{6}$$

In Table 1, we gather commonly used contact rates. It appears from the table that the faster $\beta(N)$ tends to 0, the smaller the basic reproduction number is.

TABLE 1. Commonly used contact rates

Type of contact	$\mathbf{C(N)}$	$\beta(N)\mathbf{SI}$	$\mathscr{R}_0$
Mass action	λN	λSI	$\frac{\lambda K}{\alpha+\mu+\delta}$
Standard incidence	λ	$\frac{\lambda SI}{N}$	$\frac{\lambda}{\alpha+\mu+\delta}$
Michaelis-Menten	$\frac{\lambda N}{1+bN}$	$\frac{\lambda SI}{1+bN}$	$\frac{\lambda K}{(1+bK)(\alpha+\mu+\delta)}$
Saturated contact	$\frac{\lambda N}{1+bN+\sqrt{1+2bN}}$	$\frac{\lambda SI}{1+bN+\sqrt{1+2bN}}$	$\frac{\lambda K}{(1+bK+\sqrt{1+2bK})(\alpha+\mu+\delta)}$

Remark 2 *1. Notice that the solution of (5) has the closed form* $N(t) = (N_0 - K)e^{-\mu t} + K$ *when* $\alpha = 0$. *However, for* $\mathscr{R}_0 > 1$, $\alpha > 0$ *and if Eqs (5)–(6) are used, it is in general not possible to find* E_∞ *explicitly. It is obtained implicitly from the equation*

$$f(N_\infty) := \mu K \beta(N_\infty) - \mu C(N_\infty) - \alpha\left(C(N_\infty) + (\alpha + \mu + \delta)\right) = 0,$$

to which the bisection method applies. This method is illustrated in Figure 1 where $\beta(N) = \lambda e^N/N(1+e^N)$, $K = 100$, $\mu = 0.2$, $\alpha = 0.1$. *For* $\delta = 0.2$, *so that* $\mathscr{R}_0 = 4.0625$, *we obtain the approximations* $S_\infty = 5.1613$ *and* $I_\infty = 15.8065$.

2. *On the contrary, for* $\beta(N) = \frac{\lambda}{N}$ *(standard incidence), we have the following explicit expression of the endemic equilibrium:*

$$E_\infty = \left(\frac{\mu K}{\mu\mathscr{R}_0 + \alpha(\mathscr{R}_0 - 1)}, \frac{\mu K(\mathscr{R}_0 - 1)}{\mu\mathscr{R}_0 + \alpha(\mathscr{R}_0 - 1)}\right).$$

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

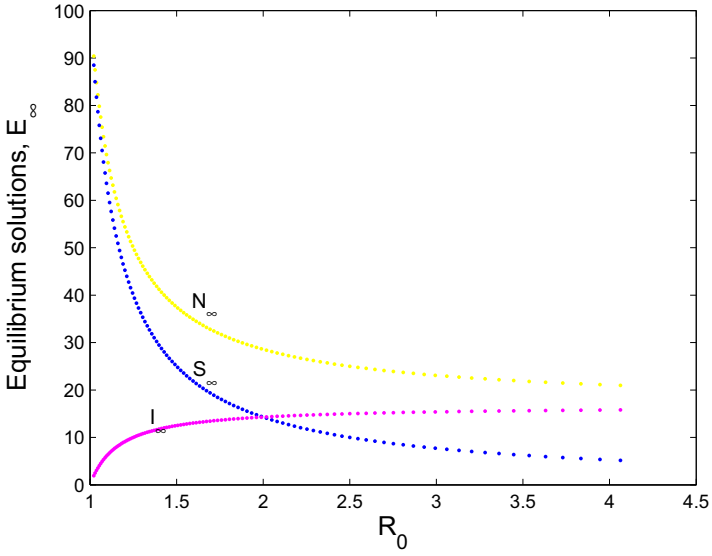


FIGURE 1. Bisection method: Plot of E_∞ against $\mathscr{R}_0$

NSFD SCHEMES FOR THE SIS MODEL

For the SIS model, we proposed the NSFD scheme

$$\frac{S_{n+1}-S_n}{\phi} = \mu K - \frac{C(N_n)I_nS_{n+1}}{S_{n+1}+I_n} - \mu S_{n+1} + \delta I_n \tag{7}$$

$$\frac{I_{n+1}-I_n}{\phi} = \frac{C(N_n)I_nS_{n+1}}{S_{n+1}+I_n} - (\alpha+\mu)I_{n+1} - \delta I_n, \tag{8}$$

where

$$\phi \equiv \phi(h) = \frac{1-e^{-\mu h}}{\mu} = h + \mathscr{O}(h^2).$$

In the scheme (7)-(8), it is important to notice the use of Mickens' rules [6, 9]: (a) The standard denominator h of the discrete derivatives is replaced by $\phi(h)$ motivated by the exact scheme of the conservation law (4); (b) The nonlinear terms in the SIS model are approximated in a nonlocal way, namely $\frac{C(N_n)I_nS_{n+1}}{S_{n+1}+I_n}$ instead of $\frac{C(N_n)I_nS_n}{S_n+I_n}$. It is precisely because of the specific nonlocal approximation adopted here that the global asymptotic stability of the disease-free fixed-point in the next theorem is easily established, as observed in [7].

Theorem 3 *The NSFD scheme (7)-(8) is dynamically consistent with Theorem 1. That is:*

1. *The NSFD scheme is a dynamical system on*

$$\Omega = \{(S,I) \in \mathbb{R}^2_+ : 0 \le S+I = N \le K\};$$

2. *If $\mathscr{R}_0 < 1$, the disease-free fixed-point $E_0 = (K,0)$ is GAS;*
3. *If $\mathscr{R}_0 > 1$, E_0 is an unstable fixed-point and $E_\infty = (S_\infty, I_\infty)$ is the unique endemic fixed-point and it is LAS;*
4. *The following discrete conservation law holds:*

$$\mu K - (\mu+\alpha)N_{n+1} \le \frac{N_{n+1}-N_n}{\phi} \le \mu(K-N_{n+1}).$$

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

In the case when the standard incidence formulation is used, the NSFD scheme reduces to:

$$\frac{S_{n+1}-S_n}{\phi} = \mu K - \frac{\lambda I_n S_{n+1}}{S_{n+1}+I_n} - \mu S_{n+1} + \delta I_n \tag{9}$$

$$\frac{I_{n+1}-I_n}{\phi} = \frac{\lambda I_n S_{n+1}}{S_{n+1}+I_n} - (\alpha+\mu) I_{n+1} - \delta I_n. \tag{10}$$

The method (9)–(10) is illustrated in Figure 2. Given the large value used here ($K = 1000$), the same figure gives an account on the NSFD scheme (7)-(8) when $\beta(N) = \lambda e^N / N(1+e^N)$.

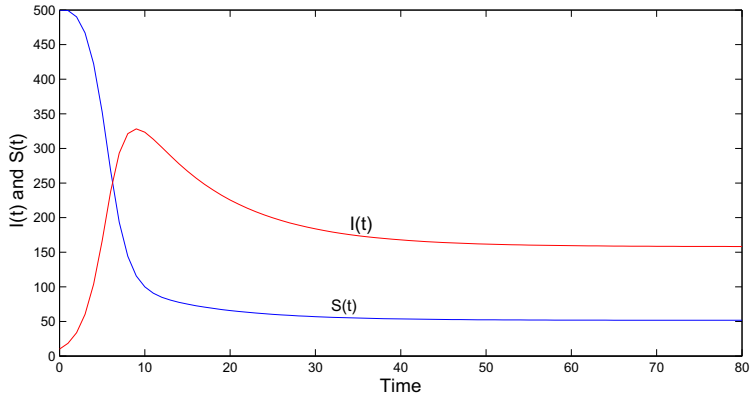


FIGURE 2. NSFD scheme (9)-(10): $K = 1000$, $\mu = 0.02$, $\alpha = 0.1$, $\delta = 0.2$, $\lambda = 1.3$, $I_0 = 10$, $\mathscr{R}_0 = 4.1$, $I_\infty = 158$ and $S_\infty = 51$

We now come to the design of a higher order NSFD scheme. To this end, let us denote by $f(x) \equiv f(S,I)$ the right hand side of the SIS model (2)–(3) and let $q > 0$ be greater or equal to the largest eigenvalue in module of the Jacobian matrices of f at the equilibria point. In view of the expression of the polynomial (see *e.g.*, [11]), which is used to define the absolute stability of the classical Runge-Kutta method of order 4, we define

$$\phi(h) := \frac{h}{1+c(qh)^4} = h + \mathscr{O}(h^5), \tag{11}$$

where $c > 0$ is any constant. We introduce the nonstandard Runge-Kutta method

$$\frac{x_{n+1}-x_n}{\phi(h)} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{12}$$

where

$$k_1 = f(x_n);\ k_2 = f(x_n + \frac{1}{2}\phi(h)k_1);\ k_3 = f(x_n + \frac{1}{2}\phi(h)k_2);\ k_4 = f(x_n + \phi(h)k_3). \tag{13}$$

The NSFD Runge-Kutta method (12)–(13) is of order 4 due to the asymptotic relation given in (11).

For numerical experiments, we assume that $N = K =$ constant and $\alpha = 0$, so that the SIS model (2)–(3) or (5)–(6), in standard incidence formulation, reduces to the following logistic equation (when $\mathscr{R}_0 > 1$) or extinction equation (when $\mathscr{R}_0 < 1$) for which the exact solution is known and $I = 0$ as well as $I = N(1 - \frac{1}{\mathscr{R}_0})$ are equilibrium points:

$$I' = (\lambda - \mu - \delta)(1 - \frac{I}{N(1-\frac{1}{\mathscr{R}_0})})I =: f(I),\ \ \mathscr{R}_0 = \frac{\lambda}{\mu+\delta}, \tag{14}$$

The largest value in module of the derivative of $f(I)$ at the equilibria being $\lambda - \mu - \delta$, we set $q = |\lambda - \mu - \delta|$ in (11). Furthermore, we replace the vector function $f(x)$ by the real-valued function $f(I)$ in (12) and (13). The excellence performance of the NS Runge-Kutta method is illustrated by the results in Table 2 as well as by Figure 3 and Figure 4. More precisely, no restriction is placed on the value of the step size h for the NS Runge-Kutta method to preserve the

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

TABLE 2. NS Runge-Kutta method with $c = 100, \mathscr{R}_0 = 0.89$

t	NSFD RK4	exact solution	Error: $h = 0.5$ $\phi = 0.398$	Error: $h = 1$ $\phi = 0.1961$	Error: $h = 2$ $\phi = 0.030$
0.0	80.000000	80.000000	1.421085e-014	1.421085e-014	1.421085e-014
18.0	23.573736	23.572750	9.854203e-004	1.036865e-002	1.576336e-002
36.0	8.620017	8.61936a	6.480276e-004	1.036865e-002	1.659326e-001
54.0	3.366977	3.366612	3.647248e-004	5.837070e-003	9.375971e-002
72.0	1.347421	1.347230	1.915396e-004	4.944246e-002	3.066153e-003
90.0	0.544358	0.544262	9.610898e-005	1.538893e-003	2.491507e-002
108	0.220757	0.220710	4.664874e-005	7.471302e-004	1.214611e-002
126	0.089662	0.089640	2.208081e-005	3.537403e-004	5.774793e-003
144	0.036440	0.036429	1.025124e-005	1.642706e-004	2.692999e-003
162	0.014813	0.014809	4.687258e-006	7.513052e-005	1.236873e-003
180	0.006022	0.006020	2.117170e-006	3.394436e-005	5.611959e-004
198	0.002449	0.002448	9.468135e-007	1.518416e-005	2.521040e-004

positivity and boundedness of solutions. Furthermore, the two figures illustrate the validity, for the NS-Runge-Kutta method, of the first three items of Theorem 3, which can be proved by using the approach in [8] for the study of the topological dynamic consistency. The fact that the standard Runge-Kutta method and other classical methods fail to replicate the positivity property is extensively discussed in the literature (see, for instance, [6, 7]). Notice that an attempt to deal with the NS Runge-Kutta method of order 2 for the logistic equation was made in [6].

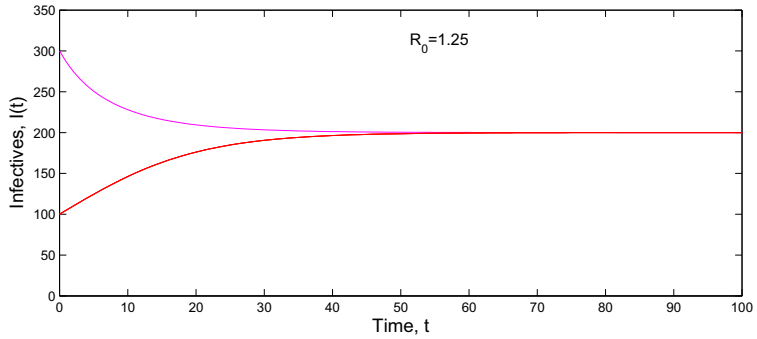


FIGURE 3. NS Runge-Kutta scheme for $I_0 = 100$, $I_0 = 300$ and $\mathscr{R}_0 = 1.25$

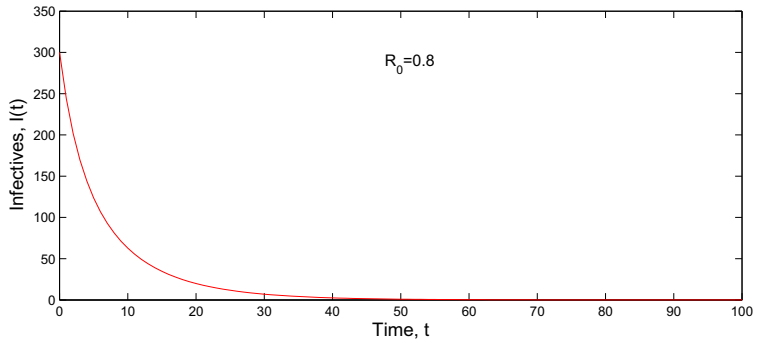


FIGURE 4. NS Runge-Kutta scheme for $I_0 = 300$ and $\mathscr{R}_0 = 0.8$

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

THE SIS-DIFFUSION MODEL

For the spread of the disease in space, we consider the the following model with standard incidence together with the assumptions $\alpha = 0$ and $\mathscr{R}_0 = \frac{\lambda}{\mu+\delta} > 1$:

$$S_t = \mu K - \frac{\lambda IS}{N} - \mu S + \delta I + S_{xx} \tag{15}$$

$$I_t = \frac{\lambda IS}{N} - (\mu+\delta)I + I_{xx}. \tag{16}$$

By adding these equations, we have the conservation law:

$$N_t = \mu K - \mu N + N_{xx}. \tag{17}$$

In Eqs. (15)–(16), we could of course consider the general contact rate $\beta(N)$ in lieu of λ/N. The choice made here is simply to easily deduce the well-posedness of the problem stated in the next two theorems from classical results in [4, 5, 12].

Theorem 4 *For $0 \le I_0(x) \le N_0(x) \le K$, the equivalent SIS-Diffusion model*

$$I_t = (\lambda - \mu - \delta)\left(1 - \frac{I}{N(1-\frac{1}{\mathscr{R}_0})}\right) I + I_{xx}, \tag{18}$$

where N satisfies the heat equation (17), admits a unique solution such that $0 \le I(x,t) \le N(x,t) \le K$.

Furthermore, the qualitative analysis of the ODE

$$W'' + cW' + (\lambda - \mu - \delta)\left(1 - \frac{\mathscr{R}_0 W}{N(\mathscr{R}_0 - 1)}\right) W = 0,$$

leads to the following result:

Theorem 5 *A Traveling Wave Solution-TWS, $I(x,t) = W(z)$, $z = x - ct \in \mathbb{R}$, to (18), exists if and only if $N = K =$ constant. In this case, we are dealing with the Fisher equation.*

NSFD SCHEME FOR THE SIS-DIFFUSION MODEL

For N constant, we consider the scheme

$$\begin{aligned} \frac{I_m^{k+1} - I_m^k}{\phi(\Delta t)} &= (\lambda - \mu - \delta)\left(1 - \frac{I_m^{k+1}}{N(1-\frac{1}{\mathscr{R}_0})}\right)\left(\frac{I_{m+1}^k + I_m^k + I_{m-1}^k}{3}\right) \\ &+ \frac{I_{m+1}^k - 2I_m^k + I_{m-1}^k}{[\psi(\Delta x)]^2}, \end{aligned}$$

which is motivated by the exact schemes [9] of the logistic equation and the harmonic oscillator that involve specific denominator functions $\phi(\Delta t)$ and $\psi(\Delta x)$, respectively. The implementation of this NSFD scheme is carried out by using its equivalent formulation:

$$I_m^{k+1} = N(1 - \frac{1}{\mathscr{R}_0}) \frac{\frac{\phi}{\psi^2}(I_{m+1}^k + I_{m-1}^k) + \phi(\lambda - \mu - \delta)\left(\frac{I_{m+1}^k + I_m^k + I_{m-1}^k}{3}\right)}{N(1-\frac{1}{\mathscr{R}_0}) + \phi(\lambda - \mu - \delta)\left(\frac{I_{m+1}^k + I_m^k + I_{m-1}^k}{3}\right)}. \tag{19}$$

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

Due to this explicit formulation and to the methodology in [13], the following result illustrated in Figure 5 (with Dirichlet boundary conditions) is readily obtained:

Theorem 6 *Under the functional relation* $\frac{\phi(\Delta t)}{\psi(\Delta x)^2} = \frac{1}{2}$ *between the step sizes, the NSFD scheme (19) replicates the positivity and boundedness properties of the exact solution. That is:*

$$0 \le I_m^k \le N(1-\frac{1}{\mathscr{R}_0}) \Rightarrow 0 \le I_m^{k+1} \le N(1-\frac{1}{\mathscr{R}_0}),$$

$$N(1-\frac{1}{\mathscr{R}_0}) \le I_m^k \le N \Rightarrow N(1-\frac{1}{\mathscr{R}_0}) \le I_m^{k+1} \le N.$$

Based on [9], we can take in Theorem 6 either

$$\phi(\Delta t) = \frac{\exp((\lambda-\mu-\delta)\Delta t)-1}{\lambda-\mu-\delta} \text{ or } \psi(\Delta x) = \frac{2\sin(\frac{(\lambda-\mu-\delta)\Delta x}{2})}{\lambda-\mu-\delta}.$$

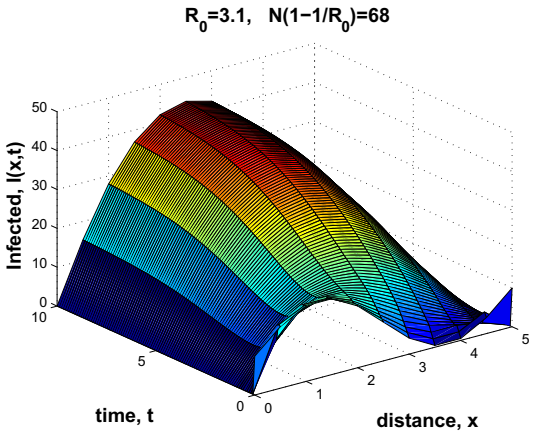


FIGURE 5. NSFD scheme (19) for $\lambda = 1.3,\ \mu = 0.2,\ \delta = 0.2,\ I_0 = 10+10\sin(2\pi x/5)$ and $N = 100$

CONCLUSION

We extended the classical SIS model in such a way that it involves a more general contact rate for the transmission of the disease in time as well as a diffusion term for the spread of the disease in the space variable. We investigated the dynamics of the model, designed NSFD schemes, which replicate these dynamics and introduced a higher order NSFD scheme of Runge-Kutta type.

Our ongoing work, based on [10], includes the extension of this study to the Volterra integral equation, with the contact rate depending now on the number of infective individuals, in order to take into account the period of infectivity.

ACKNOWLEDGMENTS

Authors were all supported by the National Research Foundation of South Africa. The authors would like to thank Prof. C.I. Christov and Prof. M.D. Todorov for the invitation to present this work at the AMiTaNS'11 conference in Albena, Bulgaria. Thanks are also addressed to the anonymous referee for relevant suggestions and remarks.

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions

REFERENCES

1. H. W. Hethcote (2000) The mathematics of infectious disease, *SIAM Review* **42**, 599–653.
2. F. Brauer, P. van den Driessche, and Jian Hong Wu (Eds), *Mathematical Epidemiology,* Springer LNM on Mathematical Biosciences Subseries, Vol. **1945**, 2008.
3. F. Brauer and C. Castillo-Chavez, *Mathematical models in population biology and epidemiology*, Springer, New York, 2001.
4. J. D. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications*, Springer, New York, 2003.
5. J. Smoller, *Shock Waves and Reaction-Diffusion Equations,* Springer, New York, 1983.
6. R. Anguelov and J.M-S. Lubuma (2001) Contributions to the mathematics of the nonstandard finite difference method and applications, *Numerical Methods for Partial Differential Equations* **17**, 518–543.
7. S. M. Garba, A. B. Gumel, and J. M.-S. Lubuma (2011) Dynamically consistent nonstandard finite difference method for an epidemic model, *Mathematical and Computer Modelling* **53**, 131–150.
8. R. Anguelov, J.M.-S. Lubuma, and M. Shillor, Topological dynamic consistency of non-standard finite difference schemes for dynamical systems, *Journal of Differnce Equations and Applications,* DOI 10.1080/10236198.2010.488226. (in Press)
9. R.E. Mickens, *Nonstandard finite difference models of differential equations,* World Scientific, Singapore, 1994.
10. P. van den Driessche and J. Watmough (2000) A simple SIS epidemic model with a backward bifurcation, *Journal of Mathematical Biology* **40**, 523–540.
11. A. Quarteroni, R. Sacco, and F. Saleri, *Numerical Mathematics,* Springer, Berlin, 2007.
12. J. D. Logan, *An Introduction to nonlinear partial differential equations*, John Wiley and Sons, New York, 1994.
13. R. Anguelov, P. Kama, and J.M.-S. Lubuma (2005) On non-standard finite difference models of reaction-diffusion equations, *Journal of Computational and Applied Mathematics*, **175**, 11–29.

Downloaded 05 Dec 2011 to 137.215.6.53. Redistribution subject to AIP license or copyright; see http://proceedings.aip.org/about/rights_permissions