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RHOADES-TYPE FIXED POINT THEOREM FOR PARTIAL METRIC SPACES

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Abstract. In this paper we obtain a fixed point theorem for non-self mappings in partial metric spaces. We make use of the technique by Assad and Kirk who derived fixed point theorems for metrically convex metric spaces. Our results generalize earlier results by Rhoades. An illustrative example is also provided.

Key words: Metrically convex, Partial metric spaces, Non-self mapping
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1. Introduction and Preliminaries

The concept of partial metric spaces was introduced by Mathews [7] in 1992 as part of the study of denotational semantics of dataflow networks. Mathews also extended the Banach contraction theorem to partial metric spaces. A number of researchers have developed fixed point theorems for self mappings in partial metric spaces.

The study of fixed points of non-self mappings was originated with the investigation of Halpern [4], where he obtained a generalization of the Schauder-Tychonoff Theorem [2]. Also, Gajić and Rakočević [3] developed a fixed point theorem for non-self mappings in metric spaces with a Takahashi convex structure. Rhoades used the concept of metrically convex spaces introduced by Assad and Kirk [1] to develop a fixed point theorem for non-self mappings for what he termed as generalized contractions in metric spaces. Rhoades theorem has been extended by various authors including Imdad and Kumar [5, 6].

In this study, we seek to extend Rhoades' theorem so that it applies to partial metric spaces.

Throughout this paper, the letters $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of all non negative real numbers and the set of all positive integer numbers respectively.

We will require the following preliminaries for our results.

Definition 1.1. [7] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:

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(P1): $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(P2): $p(x,x) \leq p(x,y)$
(P3): $p(x,y) = p(y,x)$,
(P4): $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

From (P1) and (P2) we have

$$p(x,y) = 0 \Rightarrow p(x,y) = p(x,x) = p(y,y) \Rightarrow x = y. \tag{1.1}$$

From (P4) we have

$$p(x,y) \leq p(x,z) + p(z,y). \tag{1.2}$$

An example of a partial metric space (X,p) is when $X = \mathbb{R}_+$, the set of all non-negative real numbers, and $p(x,y) = \max\{x,y\}$ for all $x,y \in X$.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where
$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. [7] Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i): $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$.
(ii): $\{x_n\}$ is called a Cauchy sequence if only if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.
iii): A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Recall the following lemma due to Mathews [7]:

Lemma 1.3. [7] *If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}_+$ given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.3}$$

defines a metric on X.

We take note of the following lemma.

Lemma 1.4. [7] *Let (X,p) be a partial metric space.*
(i) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).
(ii) (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore $\lim_{n\to+\infty} p(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.

In this study, we call p^s a metric derived from the partial metric p.

Lemma 1.5. *Let (X, p) be a partial metric space and let p^s be the metric derived from the partial metric p. Then for all $a, b, c \in X$ we have*

$$p^s(a, b) \leq p^s(a, c) \Rightarrow p(a, b) \leq 2p(a, c). \tag{1.4}$$

Proof. From Lemma 1.3 we have

$$\begin{aligned} p^s(a, b) &\leq p^s(a, c) \\ &\Rightarrow 2p(a, b) - p(a, a) - p(b, b) \leq 2p(a, c) - p(a, a) - p(c, c) \\ &\Rightarrow 2p(a, b) - p(b, b) \leq 2p(a, c) - p(c, c). \end{aligned} \tag{1.5}$$

From Lemma 1.1, P2 we have $p(b, b) \leq p(a, b)$, implying that $p(a, b) = 2p(a, b) - p(a, b) \leq 2p(a, b) - p(b, b)$. We also have $2p(a, c) - p(c, c) \geq 2p(a, c)$. Hence (1.5) becomes

$$p(a, b) \leq 2p(a, c). \tag{1.6}$$

□

The following definition of a metrically convex metric space is due to Assad and Kirk [1].

Definition 1.6. [1] A complete metric space (X, d) is said to be (metrically) convex if X has the property that for each $x, y \in X$ with $x \neq y$ there exists $z \in X, x \neq z \neq y$, such that $d(x, z) + d(z, y) = d(x, y)$.

If (X, d) is a metrically convex metric space, and $x, y \in X$, we term

$$seg[x, y] := \{z \in X : d(x, y) = d(x, z) + d(z, y)\}.$$

We get the following lemma from Assad and Kirk [1].

Lemma 1.7. *Let K be a closed subset of the complete and convex metric space X. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x, z) + d(z, y) = d(x, y).$$

We now introduce the metrically convex partial metric space.

Definition 1.8. A partial metric space (X, p) is said to be metrically convex if the corresponding metric space (X, p^s) is metrically convex in the sense of Lemma 1.3, where $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$.

As an example, the partial metric space $(\mathbb{R}_+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$ is metrically convex because (X, p^s) where $p^s(x, y) = |x - y|$ is the metric derived from the partial metric p.

Lemma 1.9. *Let (X, p) be a metrically convex partial metric space. Let $x, y \in X$. If $z \in \text{seg}[x, y]$ then:*
(i) $p(x, y) + p(z, z) = p(x, z) + p(z, y)$,
(ii) $p(x, y) \geq p(x, z)$.

Proof. Applying (1.3) to Definition 1.6, if $z \in \text{seg}[x, y]$, then we have:

$$\begin{aligned} & p^s(x,y) = p^s(x,z) + p^s(z,y) \\ \Rightarrow\ & 2p(x,y) - p(x,x) - p(y,y) = 2p(x,z) - p(x,x) - p(z,z) \\ & \qquad + 2p(z,y) - p(z,z) - p(y,y) \\ \Rightarrow\ & p(x,y) = p(x,z) + p(z,y) - p(z,z) \\ \Rightarrow\ & p(x,y) + p(z,z) = p(x,z) + p(z,y). \end{aligned}$$

As $p(z,y) - p(z,z) \geq 0$, from P2 of Definition 1.1 we have

$$p(x,y) \geq p(x,z).$$

□

We now prove the following lemma which is an extension of Lemma 1.7 into metrically convex partial metric spaces.

Lemma 1.10. *Let K be a non-empty subset of a metrically convex partial metric space (X,p) which is closed in (X,p^s). If $x \in K$ and $y \in X\backslash K$, then there exists a point $z \in \partial K$ (the boundary of K with respect to (X,p^s)) such that*

$$p(x,y) + p(z,z) = p(x,z) + p(z,y).$$

Proof. From Definition 1.8, if the partial metric space (X,p) is metrically convex, then (X,p^s) is metrically convex. From Lemma 1.7 (ii), this means that if $x \in C$ and $y \in X\backslash K$ then there exists z in ∂K, (the boundary of K), such that $p^s(x,y) = p^s(x,z) + p^s(z,y)$. Using (1.3), this means

$$\begin{aligned} & p^s(x,y) = p^s(x,z) + p^s(z,y) \\ \Rightarrow\ & 2p(x,y) - p(x,x) - p(y,y) = 2p(x,z) - p(x,x) - p(z,z) \\ & \qquad + 2p(z,y) - p(z,z) - p(y,y) \\ & \Rightarrow 2p(x,y) = 2p(x,z) + 2p(z,y) - 2p(z,z) \\ & \Rightarrow p(x,y) + p(z,z) = p(x,z) + p(z,y) \\ & \Rightarrow p(x,z) + p(z,y) = p(x,y) + p(z,z). \end{aligned}$$

□

We now prove the following lemma, which is modified from Theorem 1 of Assad and Kirk [1], and is necessary for our work.

Lemma 1.11. *Consider a sequence $\{w_n\}_{n\in\mathbb{N}} \in \mathbb{R}_+$ such that, for all $n \geq 2$ we have*

$$w_n \leq k \max\{w_{n-2}, w_{n-1}\}, k \in (0,1), \tag{1.7}$$

then

$$w_n \leq k^{n/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.8}$$

Proof. We prove the lemma by the induction. First we show that Lemma 1.11 holds for $n = 2$.

We note that $k \in (0,1)$ implies $k < k^{1/2}$. Hence if $n = 2$, then (1.7) leads to

$$w_2 \leq k \max\{w_0, w_1\} \leq k^{1/2} \max\{w_0, w_1\} = k^{2/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.9}$$

We then show that the lemma holds for $n = 3$. If $n = 3$, then (1.7) leads to $w_3 \leq k \max\{w_1, w_2\}$. If $w_1 \geq w_2$, then we get

$$\begin{aligned} & w_3 \leq k \max\{w_1, w_2\} \\ \Rightarrow\ & w_3 \leq k w_1 \\ & \leq k \max\{w_0, w_1\} \\ & = k^{3/2} \cdot k^{-1/2} \max\{w_0, w_1\}. \end{aligned}$$

If however $w_1 < w_2$, we get

$$\begin{aligned} & w_3 \leq k \max\{w_1, w_2\} \\ \Rightarrow\ & w_3 \leq k w_2 \\ \Rightarrow\ & w_3 \leq k \times k^{2/2} k^{-1/2} \max\{w_0, w_1\}, \text{ from (1.9)} \\ & \leq k^{3/2} \max\{w_0, w_1\} \\ & \leq k^{3/2} \cdot k^{-1/2} \max\{w_0, w_1\}, \text{ because } k^{-1/2} \geq 1. \end{aligned}$$

We now show that, if Lemma 1.11 holds for $1 \leq n \leq j$ where $j \geq 2$, then it must be hold for $j + 1$. Hence we have from (1.7)

$$w_{j+1} \leq k \max\{w_{j-1}, w_j\}. \tag{1.10}$$

We consider two cases.

Case (i): Suppose $w_{j-1} \leq w_j$. Then (1.10) becomes

$$\begin{aligned} w_{j+1} & \leq k w_j \\ & \leq k \cdot k^{j/2} k^{-1/2} \max\{w_0, w_1\} \text{ from (1.8)} \\ & = k^{(j+2)/2} k^{-1/2} \max\{w_0, w_1\}. \end{aligned} \tag{1.11}$$

Case (ii): Suppose $w_{j-1} > w_j$. Then (1.10) becomes

$$\begin{aligned} w_{j+2} & \leq k w_{j-1} \\ & \leq k \cdot k^{(j-1)/2} k^{-1/2} \max\{w_0, w_1\} \text{ from (1.8)} \\ & = k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}. \end{aligned} \tag{1.12}$$

We note that for $j \geq 2$ and $k \in (0,1)$ we have $k^{(j+1)/2} > k^{(j+2)/2}$. Hence (1.11) and (1.12) imply that

$$w_{j+1} \leq k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}.$$

□

Rhoades [9] proved a fixed point theorem where T was a non-self mapping. The theorem states as follows.

Theorem 1.12. *Let X be a Banach space, K a non empty closed subset of X and $T : K \to X$ be a mapping satisfying the condition*

$$d(Tx, Ty) \leq h \max\{d(x, y)/2, d(x, Tx), d(y, Ty), [d(x, Ty) + d(Tx, y)]/q\}$$

for all $x, y \in K$, $0 \leq h < 1, q \geq 1 + 2h$ and T has the additional property that for each $x \in \partial K$, the boundary of $K, Tx \in K$, then T has a unique fixed point.

In this paper we seek to extend Theorem 1.12 that was proved for metric spaces into partial metric spaces.

2. **Main Results**

We aim to prove the following theorem:

Theorem 2.1. *Let (X, p) be a complete metrically convex partial metric space and K a non empty closed subset of X, the closure being with respect to (X, p^s). Let ∂K, the boundary of K with respect to (X, p^s), be non empty. Let $T : K \to X$ be a mapping satisfying the condition*

$$p(Tx, Ty) \leq h \max\{p(x, y)/2, p(x, Tx), p(y, Ty), [p(x, Ty) + p(Tx, y)]/q\}$$

for all $x, y \in K$, $0 \leq 2h < 1, q \geq 1 + 2h$. Furthermore let T have the following additional properties:
(i) $x \in \partial K \Rightarrow Tx \in K$ and
(ii) TK is closed in (X, p^s).
Then T has a unique fixed point z in X, and $p(z, z) = 0$.

Proof. We choose $x_0 \in \partial K$. From assumption (i) we can find $x_1 \in K$ such that $x_1 = Tx_0$. We then find Tx_1. If $Tx_1 \in K$, we set $x_2 = Tx_1$. Otherwise if $Tx_1 \notin K$, we choose $x_2 \in \partial K$ such that

$$p(x_1, Tx_1) + p(x_2, x_2) = p(x_1, x_2) + p(x_2, Tx_1)$$

as per Lemma 1.10.

In general we generate the sequence $\{x_n\}$ in the following way. We commence with $x_0 \in \partial K$ and continue inductively as follows:

If $Tx_n \in K$, then $x_{n+1} = Tx_n$. Otherwise if $Tx_n \notin K$, as per Lemma 1.10, we choose $x_{n+1} \in \partial K$ such that

$$p(x_n, Tx_n) + p(x_{n+1}, x_{n+1}) = p(x_n, x_{n+1}) + p(x_{n+1}, Tx_n).$$

Let us define two sets, P and Q forming a partition of the set $\{x : x = x_n, n \in \mathbb{N}\}$. Let $P = \{x_n \in \{x_n\} : x_n = Tx_{n-1}, n \geq 1\}$ and $Q = \{x_n \in \{x_n\} : x_n \neq Tx_{n-1}\}$. Note that $x_n \in Q$ implies that $x_n \in \partial K$ and $x_n \in \text{seg}[x_{n-1}, Tx_{n-1}]$.

We investigate three cases.

Case 1: Let $(x_n, x_{n+1}) \in P \times P$.

Then for $n \geq 1$, using assumption (i) we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\leq h \max\{p(x_{n-1}, x_n)/2, p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \\ &\qquad [p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})]/q\} \\ &= h \max\{p(x_{n-1}, x_n)/2, p(x_{n-1}, x_n), p(x_n, x_{n+1}), \\ &\qquad [p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]/q\}. \end{aligned}$$

From P4 of Definition 1.1, we have

$$p(x_{n-1}, x_{n+1}) + p(x_n, x_n) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}).$$

Hence

$$p(x_n, x_{n+1}) \leq h \max\{p(x_{n-1}, x_n), [p(x_{n-1}, x_n) + p(x_n, x_{n+1})]/q\}.$$

If $p(x_{n-1}, x_n) \geq [p(x_{n-1}, x_n) + p(x_n, x_{n+1})]/q$, then we have

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n).$$

Otherwise if $p(x_{n-1}, x_n) < [p(x_{n-1}, x_n) + p(x_n, x_{n+1})]/q$, we have

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \frac{h}{q}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})] \\ &\leq \frac{h}{q-h} p(x_{n-1}, x_n). \end{aligned}$$

From the conditions in the assumption, we have

$$q \geq (1 + 2h) \Rightarrow (q - h) \geq (1 + h) \Rightarrow 1 \geq \frac{1}{q-h} \Rightarrow h \geq \frac{h}{q-h}. \tag{2.1}$$

Hence we have

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n).$$

In either situation, we have for $(x_n, x_{n+1}) \in P \times P$,

$$p(x_n, x_{n+1}) \leq hp(x_{n-1}, x_n).$$

Case 2: Let $(x_n, x_{n+1}) \in P \times Q$.

Then from Lemma 1.9, as $x_{n+1} \in \text{seg}[x_n, Tx_n]$, we have

$$p(x_n, x_{n+1}) \leq p(x_n, Tx_n) = p(Tx_{n-1}, Tx_n) \leq hp(x_{n-1}, x_n),$$

using an argument similar to that of Case 1.

Case 3: Let $(x_n, x_{n+1}) \in Q \times P$. We claim that for $n \geq 1$, $x_n \in Q$ implies $x_{n-1} \in P$.

Let $x_{n-1} \in Q$, then $x_{n-1} \in \partial K$. This means, from (i) in the assumption, that $Tx_{n-1} \in K$, and hence $x_n \in P$ which is a contradiction. implying $x_{n-1} \in P$ for $n \geq 1$.

Because $x_n \in Q$ implies $x_n \in \text{seg}[x_{n-1}, Tx_{n-1}]$, we have

$$p^s(x_n, x_{n+1}) \leq \max\{p^s(x_{n-1}, x_{n+1}), p^s(Tx_{n-1}, x_{n+1})\}, \tag{2.2}$$

(see Imdad and Kumar [5]).

We consider two scenarios. If $p^s(x_{n-1}, x_{n+1}) \leq p^s(Tx_{n-1}, x_{n+1})$, then (2.2) becomes

$$\begin{aligned}
p^s(x_n, x_{n+1}) &\leq p^s(Tx_{n-1}, x_{n+1}) \\
\Rightarrow p(x_n, x_{n+1}) &\leq 2p(Tx_{n-1}, x_{n+1}), \text{ as per Lemma 1.5} \\
&= 2p(Tx_{n-1}, Tx_n) \\
&\leq 2h \max\{p(x_{n-1}, x_n)/2, p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \\
&\qquad [p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1}]/q\} \\
&= 2h \max\{p(x_{n-1}, x_n)/2, p(x_{n-1}, Tx_{n-1}), p(x_n, x_{n+1}), \\
&\qquad [p(x_{n-1}, x_{n+1}) + p(x_n, Tx_{n-1}]/q\} \\
&= 2h \max\{p(x_{n-1}, Tx_{n-1}), [p(x_{n-1}, x_{n+1}) + p(x_n, Tx_{n-1}]/q\}. \qquad (2.3)
\end{aligned}$$

The last step is because $x_n \in \text{seg}[x_{n-1}, Tx_{n-1}]$, which implies by Lemma 1.9 (ii) that $p(x_{n-1}, Tx_{n-1}) \geq p(x_{n-1}, x_n) \geq p(x_{n-1}, x_n)/2$.

Using P4 of Definition 1.1, we have

$$\begin{aligned}
&p(x_{n-1}, x_{n+1}) + p(x_n, Tx_{n-1}) \\
&\qquad \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, Tx_{n-1}).
\end{aligned}$$

From Lemma 1.9 (i), we have
$p(x_{n-1}, x_n) + p(x_n, Tx_{n-1}) - p(x_n, x_n) = p(x_{n-1}, Tx_{n-1})$. Hence (2.3) becomes

$$p(x_n, x_{n+1}) \leq 2h \max\{p(x_{n-1}, Tx_{n-1}), [p(x_{n-1}, Tx_{n-1}) + p(x_n, x_{n+1})]/q\}. \qquad (2.4)$$

If $p(x_{n-1}, Tx_{n-1}) \geq [p(x_{n-1}, Tx_{n-1}) + p(x_n, x_{n+1})]/q$, (2.4) becomes

$$\begin{aligned}
p(x_n, x_{n+1}) &\leq 2hp(x_{n-1}, Tx_{n-1}) \\
&= 2hp(Tx_{n-2}, Tx_{n-1}) \\
&\leq 2h \cdot hp(x_{n-2}, x_{n-1}), \text{ according to Case 2} \\
&< hp(x_{n-2}, x_{n-1}), \qquad (2.5)
\end{aligned}$$

because of the condition $2h < 1$ in the assumption.

If otherwise $p(x_{n-1}, Tx_{n-1}) < [p(x_{n-1}, Tx_{n-1}) + p(x_n, x_{n+1})]/q$, then (2.4) becomes

$$\begin{aligned}
p(x_n, x_{n+1}) &\leq \frac{2h}{q}[p(x_{n-1}, Tx_{n-1}) + p(x_n, x_{n+1})] \\
\Rightarrow p(x_n, x_{n+1}) &\leq \frac{2h}{(q-2h)} p(x_{n-1}, Tx_{n-1}).
\end{aligned}$$

We note that

$$q \geq 2h + 1 \Rightarrow q - 2h \geq 1 \Rightarrow 2h \geq \frac{2h}{q-2h}.$$

We also note from the assumption that $2h < 1$. Thus

$$\begin{aligned} p(x_n, x_{n+1}) &\leq 2hp(x_{n-1}, Tx_{n-1}) \\ &\leq 2hp(Tx_{n-2}, Tx_{n-1}) \\ &\leq 2h \cdot hp(x_{n-2}, x_{n-1}) \\ &< hp(x_{n-2}, x_{n-1}). \end{aligned} \tag{2.6}$$

We now consider the scenario where $p^s(x_{n-1}, x_{n+1}) \geq p^s(Tx_{n-1}, x_{n+1})$.

Then (2.2) becomes $p^s(x_n, x_{n+1}) \leq p^s(x_{n-1}, x_{n+1})$, which implies from Lemma 1.5 that

$$\begin{aligned} p(x_n, x_{n+1}) &\leq 2p(x_{n-1}, x_{n+1}) \qquad (2.7) \\ &= 2p(Tx_{n-2}, Tx_n) \\ &\leq 2h \max\{p(x_{n-2}, x_n)/2, p(x_{n-2}, Tx_{n-2}), p(x_n, Tx_n), \\ &\qquad [p(x_{n-2}, Tx_n) + p(Tx_{n-2}, x_n)]/q\} \\ &= 2h \max\{p(x_{n-2}, x_n)/2, p(x_{n-2}, x_{n-1}), p(x_n, x_{n+1}), \\ &\qquad [p(x_{n-2}, x_{n+1}) + p(x_{n-1}, x_n)]/q\} \\ &= 2h \max\{p(x_{n-2}, x_n)/2, p(x_{n-2}, x_{n-1}), \\ &\qquad [p(x_{n-2}, x_{n+1}) + p(x_{n-1}, x_n)]/q\}. \qquad (2.8) \end{aligned}$$

From (1.2), we have

$$\begin{aligned} p(x_{n-2}, x_n)/2 &\leq [p(x_{n-2}, x_{n-1}) + p(x_{n-1}, x_n)]/2 \\ &\leq \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\} \\ &\leq \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, Tx_{n-1})\}, \end{aligned}$$

because $p(x_{n-1}, Tx_{n-1}) \geq p(x_{n-1}, x_n)$, as per Lemma 1.9 (ii).

Thus we may replace $p(x_{n-2}, x_n)/2$ by
$\max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, Tx_{n-1})\}$ in the right-hand side of (2.8). Hence we have

$$\begin{aligned} p(x_n, x_{n+1}) \leq 2h \max\{&p(x_{n-2}, x_{n-1}), p(x_{n-1}, Tx_{n-1}), \\ &[p(x_{n-2}, x_{n+1}) + p(x_{n-1}, x_n)]/q\}. \end{aligned} \tag{2.9}$$

If the maximum in the right-hand side of (2.9) is $p(x_{n-2}, x_{n-1})$, we have

$$p(x_n, x_{n+1}) \leq 2hp(x_{n-2}, x_{n-1}). \tag{2.10}$$

If the maximum in the right-hand side of (2.9) is $p(x_{n-1}, Tx_{n-1})$, and noting that $2h < 1$, we have

$$\begin{aligned} p(x_n, x_{n+1}) &\leq 2hp(x_{n-1}, Tx_{n-1}) \\ &= 2hp(Tx_{n-2}, Tx_{n-1}) \\ &\leq 2h \cdot hp(x_{n-2}, x_{n-1}) \\ &< hp(x_{n-2}, x_{n-1}). \end{aligned}$$

If the maximum on the right-hand side of (2.9) is $[p(x_{n-2}, x_{n+1}) + p(x_{n-1}, x_n)]/q$, we have

$$\begin{aligned} p(x_n, x_{n+1}) &\leq 2p(x_{n-1}, x_{n+1}), \text{ see (2.7)} \\ &\leq \frac{2h}{q}[p(x_{n-2}, x_{n+1}) + p(x_{n-1}, x_n)] \\ &\leq \frac{2h}{q}[p(x_{n-2}, x_{n+1}) + p(x_{n-1}, Tx_{n-1})] \\ &= \frac{2h}{q}[p(x_{n-2}, x_{n+1}) + p(Tx_{n-2}, Tx_{n-1})] \\ &\leq \frac{2h}{q}[p(x_{n-2}, x_{n+1}) + hp(x_{n-2}, x_{n-1})]. \end{aligned}$$

We expand $p(x_{n-2}, x_{n+1})$ using (1.2). We get

$$\begin{aligned} p(x_n, x_{n+1}) &\leq 2p(x_{n-1}, x_{n+1}) \\ &\leq \frac{2h}{q}[p(x_{n-2}, x_{n-1}) + p(x_{n-1}, x_{n+1}) + hp(x_{n-2}, x_{n-1})] \\ &\leq \frac{2h}{q}[(1+h)p(x_{n-2}, x_{n-1}) + p(x_{n-1}, x_{n+1})] \\ &\leq \frac{2h(1+h)}{q-h}p(x_{n-2}, x_{n-1}). \end{aligned} \tag{2.11}$$

We note that

$$q \geq 1 + 2h \Rightarrow q - h \geq 1 + h \Rightarrow 1 \geq \frac{1+h}{q-h} \Rightarrow h \geq \frac{h(1+h)}{q-h}.$$

Hence (2.11) becomes

$$p(x_n, x_{n+1}) \leq 2hp(x_{n-2}, x_{n-1}). \tag{2.12}$$

Thus in all cases, Case 1, Case 2 and Case 3, we have

$$p(x_n, x_{n+1}) \leq 2h \max\{p(x_{n-2}, x_{n-1}), p(x_{n-1}, x_n)\}. \tag{2.13}$$

Applying Lemma 1.11 with $w_n = p(x_n, x_{n+1})$ and $k = 2h$ we get

$$p(x_n, x_{n+1}) \leq (2h)^{n/2}\delta,$$

where $\delta = (2h)^{1/2} \max\{p(x_0, x_1), p(x_1, x_2)\}$.

Let $m, n \in \mathbb{N}$, $n > m$. From P4 of Definition 1.1, we have

$$\begin{aligned} p(x_m, x_n) &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) - \sum_{i=m+1}^{n-1} p(x_i, x_i) \\ &\leq \sum_{i=m}^{n-1} p(x_i, x_{i+1}) \\ &\leq \sum_{i=m}^{n-1} (2h)^{i/2}\delta \\ &= \delta(2h)^{m/2}\frac{1-(2h)^{(n-m)/2}}{1-(2h)^{1/2}} \\ &\leq \delta(2h)^{m/2}\frac{1}{1-(2h)^{1/2}}. \end{aligned}$$

Taking $m, n \to +\infty$, we get

$$\lim_{m,n\to+\infty} p(x_m, x_n) = 0 < +\infty.$$

This means $\{x_n\}$ is a Cauchy sequence in K.

Now we apply Lemma 1.4. The metric space (X, p^s) is complete because, from the assumption, the partial metric space (X, p) is complete. As K is a closed subset of the complete space (X, p^s), K is complete in (X, p^s) and hence is complete in (X, p). Because $\{x_n\} \in K$, and K is complete, this means that there is $z \in K$ such that

$$\lim_{m,n\to+\infty} p(x_m, x_n) = \lim_{m,n\to+\infty} p(x_n, z) = p(z, z) = 0.$$

We now show that z is a fixed point of T.

Consider the subsequence $\{x_{n_k}\}$ of $\{x_n\}$, for which each term is in P. This implies $x_{n_k} = Tx_{n_k-1} \in K$. Considering that TK is closed in (X, p^s) and hence complete in (X, p) and taking $k \to \infty$ we get

$$\lim_{k\to\infty} x_{n_k} = z = \lim_{k\to\infty} Tx_{n_k},$$

implying $Tx_n \to z$.

Consider $u \in TK$ such that $u = Tz$. By the assumption, we have

$$\begin{aligned} p(u, Tx_{n_k-1}) &= p(Tz, Tx_{n_k-1}) \\ &\leq h\max\{p(z, x_{n_k-1})/2, p(u, Tz), p(x_{n_k-1}, Tx_{n_k-1}), \\ &\qquad [p(z, Tx_{n_k-1}) + p(x_{n_k-1}, Tz)]/q\}. \end{aligned}$$

We take $k \to +\infty$, and note that from P1 of Definition 1.1, we have $p(z, z) \leq p(u, z)$ and $p(u, u) \leq p(u, z)$. Hence, we have

$$\begin{aligned} p(u, z) &\leq h\max\{p(z, z)/2, p(u, u), p(z, z), [p(z, z) + p(z, u)]/q\} \\ &= \frac{h}{q}[p(z, z) + p(u, z)] \\ &\leq p(z, z). \end{aligned}$$

As $\dfrac{h}{q-h} < 1$, this implies

$$p(u,z) = p(Tz,z) = 0 \Rightarrow Tz = z,$$

according to (1.1), showing that z is a fixed point of T.

We now claim that z is unique. Suppose that there is $z' \in K$ such that $Tz' = z'$. Then

$$\begin{aligned} p(z,z') &= p(Tz,Tz') \\ &\leq h\max\{p(z,z')/2, p(z,Tz), p(z',Tz') + [p(z,Tz') + p(z',Tz)]/q\} \\ &= \frac{h}{q}[p(z,z') + p(z',z)] \\ &= \frac{2h}{q}p(z,z'). \end{aligned}$$

From the assumption, we have

$$q \geq 2h+1 \Rightarrow \frac{2h+1}{q} \leq 1 \Rightarrow \frac{2h}{q} < 1.$$

Hence $p(z,z') = 0$, implying from (1.1) that $z = z'$.

We have completed the proof of Theorem 2.1. □

Corollary 2.2. *If we replace the partial metric p by a metric d then equation (1.4) which states that*

$$p^s(a,b) \leq p^s(a,c) \Rightarrow p(a,b) < 2p(a,c),$$

becomes

$$p^s(a,b) \leq p^s(a,c) \Rightarrow d(a,b) < d(a,c)$$

With this property, in Theorem 2.1 we can replace the condition $0 \leq 2h < 1$ *with* $0 \leq h < 1$ *which leads to the theorem proved in Rhoades [9] as a corollary.*

Now we provide an example on the use of our result (Theorem 2.1).

Example:

Let (X,p) be a partial metric space with $X = \mathbb{R}_+$ and the partial metric $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. The metric derived from this partial metric is

$$p^s(x,y) = |x-y|.$$

This metric is metrically convex in $\mathbb{R}_+$. Let $K = [0,5] \cup [10,30]$. Let us define $T : K \to X$ as $Tx = \dfrac{x}{3}$ for all $x \in K$.

From the given information, $TK = [0,5/3] \cup [10/3,10]$ is closed with respect to (X,p^s). This makes TK is complete in (X,p^s) and hence complete in (X,p).

For $x \in \partial K = \{0,5,10,30\}$, we have $Tx = \{0,5/3,10/3,10\} \subset K$. We note that for some values $k \in K$, (such as $k = 18$), we have $Tk \notin K$.

Without loss of generality, suppose $y \geq x$. Thus

$$\begin{aligned} p(Tx, Ty) &= p(x/3, y/3) \\ &= \max\{x/3, y/3\} \\ &= y/3 \\ &\leq 0.4y \\ &\leq 0.4 \max\{y, y/3\} \\ &= 0.4p(y, Ty) \\ &\leq h \max\{p(x, y)/2, p(x, Tx), p(y, Ty), [p(x, Ty) + p(Tx, y)]/q\}, \end{aligned}$$

where $h = 0.4 \leq 1/2$.

Thus all conditions of Theorem 2.1 are satisfied and 0 is the unique fixed point of T. Furthermore $p(0, 0) = 0$.

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