COMMON FIXED POINTS FOR FOUR NON-SELF-MAPPINGS IN PARTIAL METRIC SPACES

Terentius Rugumisa, Santosh Kumar and Mohammad Imdad

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Abstract. In this paper, we formulate a common fixed point theorem for four non-self mappings in convex partial metric spaces. The result extends a fixed point theorem by Gajić and Rakočević [Pair of non-self mappings and common fixed points. Appl. Math. Comp. **187** (2007), 999-1006] proved for two non-self mappings in metric spaces with a Takahashi convex structure. We also provide an illustrative example on the use of the theorem.

Keywords: Common fixed point, convex partial metric space, non-self mapping.

MSC 2010: 47H10, 54H25

1. Introduction and Preliminaries

Gajić and Rakočević [5] proved a common fixed point theorem for non-self mappings on a Takahashi convex metric space for a pair of mappings. In their work, they generalized the theorems by Jungck [7], Das and Naik [4], Ćirić *et al.* [3], Ćirić [2] and Imdad and Kumar [6]. In this study, we extend the theorem by Gajić and Rakočević to apply for two pairs of non-self mappings in convex partial metric spaces.

We now introduce those results which will be of use in this paper.

Definition 1.1. [8] *A partial metric on a non-empty set X is a mapping $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,*
(P0): $0 \leq p(x, x) \leq p(x, y)$,
(P1): $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(P2): $p(x, y) = p(y, x)$ and
(P3): $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A pair (X, p) is said to be a partial metric space.

From Definition 1.1, we deduce that for all x, y, z in a partial metric space (X, p), we have:

$$\text{(i)} \quad p(x,y) = 0 \text{ implies } x = y, \tag{1.1}$$

$$\text{(ii)} \quad p(x,y) \leq p(x,z) + p(z,y). \tag{1.2}$$

Proof. If $p(x,y) = 0$, then $p(x,x) = 0$ because $0 \leq p(x,x) \leq p(x,y)$ from (P0). Similarly, $p(x,y) = 0$ implies $p(y,y) = 0$ because $0 \leq p(y,y) \leq p(x,y)$. Hence $p(x,y) = 0$ implies $p(x,x) = p(x,y) = p(y,y) = 0$. From (P1) this means that $x = y$.

From (P3), we infer that

$$p(x,y) \leq p(x,z) + p(z,y).$$

□

As an example, let $X = \mathbb{R}^+$ and let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x,x_n).$$

Definition 1.2. [8] *Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$,

(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to+\infty} p(x_n,x_m)$ exists and finite,

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that

$$p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Lemma 1.3. [8] *If p is a partial metric on X, then the mapping $p^s : X \times X \to [0,+\infty)$ given by*

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric.

In this paper we will denote p^s as the metric derived from the partial metric p.

Lemma 1.4. [8] *Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s),

(ii) (X, p) is complete if and only if (X, p^s) is complete. Furthermore, $\lim_{n\to\infty} p(x_n, x)$ is zero if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

We define 0-complete partial metric spaces.

Definition 1.5. [8] *Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then*

(i) the sequence $\{x_n\}$ is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$,

(ii) (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

We define a convex partial metric space.

Definition 1.6. [10] *Let (X, p) be a partial metric space and $I = [0, 1]$ be the closed unit interval. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all $(x, y, t) \in X \times X \times I$,*

$$p\left(u, W(x, y, t)\right) \leq tp(u, x) + (1 - t)p(u, y)$$

for every $u \in X$. A partial metric space (X, p), together with the convex structure W, is called a convex partial metric space.

If (X, p) is a convex partial metric space, then for every $x, y \in X$, we define

$$\text{seg}[x, y] := \{W(x, y, t) : t \in [0, 1]\}. \tag{1.3}$$

In this study, we will use the following properties for a convex partial metric space with convex structure W.

Lemma 1.7. *Let $x, y \in X$ where (X, p) is a convex partial metric space with convex structure W. Let $w \in \text{seg}[x, y]$. Then for all $u \in X$, we have*

(i) $p(u, w) \leq \max\{p(u, x), p(u, y)\}$,

(ii) $p(x, w) \leq p(x, y)$.

Proof. Suppose $\Gamma = \max\{p(u, x), p(u, y)\}$. Applying Definition 1.6, we have

$$\begin{aligned} p(u, w) &\leq tp(u, x) + (1 - t)p(u, y) \\ &\leq t\Gamma + (1 - t)\Gamma \\ &= \Gamma \\ &= \max\{p(u, x), p(u, y)\}. \end{aligned}$$

We have proved Lemma 1.7 (i). Now let us set $x = u$ in Lemma 1.7 (i). We get

$$\begin{aligned} p(x, w) &\leq \max\{p(x, x), p(x, y)\} \\ &= p(x, y), \text{ from P0 of Definition 1.1.} \end{aligned}$$

□

Definition 1.8. [1] *Let (X, p) be a partial metric space and $B \subseteq X$. Then*

(i) *B is said to be* bounded *if there is a positive number M such that $p(x, y) \leq M$ for all $x, y \in B$,*

(ii) *if B is a* bounded *set, the diameter of B is defined as*

$$\text{diam}(B) = \sup_{u,v \in B} \{p(u, v)\}.$$

Let $f : C \to X$ be a mapping, where $C \subseteq X$. We say that f is a *self mapping* if $C = X$, otherwise f is called a *non-self mapping*. If there is an element $x \in C$ such that $fx = x$, we say that x is a *fixed point* of f in X.

Suppose we have two mappings $f, g : C \to X$, with $C \subseteq X$. Let there be $x \in C$ such that $fx = gx = w$. We say that x is a *coincidence point* of f and g in X. If $x = w$, then we call x a *common fixed point* of f and g in X.

Suppose we have two mappings $f, g : C \to X$ with $C \subseteq X$. We say f and g are *coincidentally commuting* if for all $x \in C$, we have

$$fx = gx \Rightarrow fgx = gfx.$$

In this paper, we aim to extend the following theorem by Gajić and Rakočević [5] which proves the existence of a common fixed point for non-self mappings in context of metric spaces under specified conditions.

Theorem 1.9. [5] *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let C be a non-empty closed subset of X and ∂C be the boundary of C. Let $f, g : C \to X$ and suppose $\partial C \neq \emptyset$. Let us assume that f and g satisfy the following conditions:*

(i) *For every $x, y \in C, d(gx, gy) \leq M_\omega(x, y)$ where $M_\omega(x, y) = \max\{\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)], \omega_4[d(fx, gy)], \omega_5[d(gx, fy]\}$, $\omega_i : [0, +\infty) \to [0, +\infty), i \in \{1, 2, 3, 4, 5\}$ is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $\lim_{r\to\infty}[r - \omega_i(r)] = +\infty$,*

(ii) *$\partial C \subseteq f(C)$,*

(iii) *$g(C) \cap C \subset f(C)$,*

(iv) *$fx \in \partial C \Rightarrow gx \in C$ and*

(v) *$f(C)$ is closed in X.*

Then there exists a coincidence point v in C. Moreover, if $\{f, g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g.

We now proceed to the main results.

2. Main Results

In this section, we extend Theorem 1.9 to two pairs of non-self mappings. We prove the following assumption:

Theorem 2.1. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a closed subset of X with a non-empty boundary ∂C. Let*
$S, T, A, B : C \to X$. Let us assume that S, T, A and B satisfy the following conditions:

(i) For every $x, y \in C, p(Ax, By) \leq M_\omega(x, y)$ where
$M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)], \omega_4[p(Ax, Ty)], \omega_5[p(Sx, By)]\}$, *$\omega_i : [0, +\infty) \to [0, +\infty)$, $i = 1, 2, 3, 4, 5$, is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r/2$ for $r > 0$, and* $\lim_{r\to+\infty}[r - 2\omega_i(r)] = +\infty$.
(ii) $\partial C \subseteq TC, \partial C \subseteq SC$,
(iii) $Sx \in \partial C \Rightarrow Ax \in C; Tx \in \partial C \Rightarrow Bx \in C$,
(iv) $AC \cap C \subset TC, BC \cap C \subset SC$ and
(v) SC, TC are closed in C.

Then there exists a coincidence point $z \in C$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Proof. Commencing with an arbitrary point $w \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows:

From assumption (ii), there is a point $x_0 \in C$ such that $Sx_0 = w$. We find Ax_0. Then we proceed inductively as follows.

If $Ax_{2n} \in C$, then, by (iv), we choose $x_{2n+1} \in C$ such that $Tx_{2n+1} = Ax_{2n}$.

If however $Ax_{2n} \notin C$, because W is continuous in the third variable, it means that, by (iii), there is $\lambda_{2n,2n} \in (0, 1)$ such that

$$W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in \partial C.$$

By (ii), this means we can choose $x_{n+1} \in C$ such that

$$Tx_{n+1} = W(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}) \in \partial C.$$

We then determine Bx_{2n+1}.

If $Bx_{2n+1} \in C$, then, by (iv), we choose $x_{2n+2} \in C$ such that $Sx_{2n+2} = Bx_{2n+1}$.

However if $Bx_{2n+1} \notin C$, because W is continuous in the third variable, this means there is $\lambda_{2n+1,2n+1} \in (0,1)$ such that

$$W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$

By (ii), this means we can choose $x_{n+2} \in C$ such that

$$Sx_{n+2} = W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$

We then determine Ax_{2n+2}.

We show that, for $n \geq 1$, we have

$$Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_{2n}. \tag{2.1}$$

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \partial C$, which by (iii) means $Ax_{2n} \in C$. This, by (iv), implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction. Using a similar argument we have

$$Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1}. \tag{2.2}$$

We now prove that the sequences $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geq 1$ let

$$D_n = \left(\bigcup_{i=0}^{n-1} \{Ax_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Sx_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Tx_{2i+1}\}\right),$$

$n \geq 1$.

Let $\alpha_n = \operatorname{diam}(D_n)$. We show that

$$\alpha_n \leq \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1}\}, 0 \leq j \leq n-1. \tag{2.3}$$

Let us consider the case where $\alpha_n = 0, n \geq 1$.

This means $p(Sx_0, Ax_0) = p(Ax_0, Bx_1) = p(Bx_1, Tx_1) = 0$. Applying (1.2) this means

$$Sx_0 = Ax_0 = Bx_1 = Tx_1.$$

We shall show that Sx_0 is a common fixed point of S and A. As the mappings S and A are coincidentally commuting at the coincidence point x_0, we have

$$Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0. \tag{2.4}$$

From (i), we have

$$\begin{aligned}
p(SSx_0, Sx_0) &= p(ASx_0, Bx_1) \leq M_\omega(Sx_0, x_1) \\
&= \max\{\omega_1[p(SSx_0, Tx_1)], \omega_2[p(ASx_0, SSx_0)], \omega_3[p(Bx_1, Tx_1)], \\
&\qquad \omega_4[p(ASx_0, Tx_1)], \omega_5[p(SSx_0, Bx_1)]\} \\
&= \max\{\omega_1[p(SSx_0, Sx_0)], \omega_2[p(SSx_0, SSx_0)], \omega_3[p(Sx_0, Sx_0)], \\
&\qquad \omega_4[p(SSx_0, Sx_0)], \omega_5[p(SSx_0, Sx_0)]\} \\
&\leq \omega_t[p(SSx_0, Sx_0)], \text{ for some } t \in \{1, 2, 3, 4, 5\}, \\
&< \frac{1}{2}p(SSx_0, Sx_0), \text{ for } p(SSx_0, Sx_0) > 0 \\
&\Rightarrow p(SSx_0, Sx_0) = 0.
\end{aligned}$$

By (1.2), this implies

$$SSx_0 = Sx_0. \tag{2.5}$$

Hence Sx_0 is a fixed point of S. From (2.4) we have $SSx_0 = ASx_0$. Thus (2.5) implies $ASx_0 = Sx_0$, making Sx_0 a fixed point of A too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of T and B. Hence, $z = Sx_0$ is a common fixed point of all four mappings S, T, A and B.

To show the uniqueness of the fixed point, let z' be also a fixed point of S, T, A and B. Then we have

$$\begin{aligned}
p(z, z') &= p(Az, Bz') \\
&\leq \max\{\omega_1[p(Sz, Tz')], \omega_2[p(Az, Sz)], \omega_3[p(Bz', Tz')], \\
&\qquad \omega_4[p(Az, Tz')], \omega_5[p(Sz, Bz')]\} \\
&= \max\{\omega_1[p(z, z')], \omega_2[p(z, z)], \omega_3[p(z', z')], \\
&\qquad \omega_4[p(z, z')], \omega_5[p(z, z')]\} \\
&\leq \omega_i[p(z, z')] \text{ for some } i \in \{1, 2, 3, 4, 5\} \\
&< \frac{1}{2}p(z, z'), \text{ for } p(z, z') > 0, \\
&\Rightarrow p(z, z') = 0 \\
&\Rightarrow z = z, \text{ by (1.1)}.
\end{aligned}$$

Hence when $\alpha_n = 0$, $z = Sx_0$ is the unique common fixed point of S, T, A and B.

We now consider the cases when $\alpha_n > 0$.

Case 1: Consider the case where $\alpha_n = p(Sx_{2i}, Ax_{2j})$ for some $0 \leq i, j \leq n - 1$.

(1. i) If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ we have

$$\begin{aligned}\alpha_n &= p(Sx_{2i}, Ax_{2j}) = p(Ax_{2j}, Bx_{2i-1}) \\ &\leq M_\omega(x_{2j}, x_{2i-1}) \\ &\leq \omega_s(\alpha_n) \text{ for some } s \in \{1,2,3,4,5\} \\ &< \frac{1}{2}\alpha_n,\end{aligned}$$

which is a contradiction. Hence $i = 0$.

(1.ii) If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it implies $Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and hence, by Lemma 1.7 (i), we have

$$\alpha_n = p(Sx_{2i}, Ax_{2j}) \leq \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i-2}, Ax_{2j})\}.$$

(1.ii.1) If $p(Ax_{2j}, Bx_{2i-1}) \geq p(Ax_{2i-2}, Ax_{2j})$, we have

$$\alpha_n = p(Sx_{2i}, Ax_{2j}) \leq p(Ax_{2j}, Bx_{2i-1}),$$

which leads to the contradiction in (1.i), meaning that $i = 0$.

(1.ii.2) Otherwise, if $p(Ax_{2j}, Bx_{2i-1}) < p(Ax_{2i-2}, Ax_{2j})$, then for some k such that $2i - 2 < 2k + 1 < 2j$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned}\alpha_n &= p(Sx_{2i}, Ax_{2j}) \leq p(Ax_{2i-2}, Ax_{2j}) \\ &\leq p(Ax_{2i-2}, Bx_{2k+1}) + p(Ax_{2j}, Bx_{2k+1}), \text{ by (1.2)} \\ &\leq M_\omega(x_{2i-2}, x_{2i-1}) + M_\omega(x_{2j}, x_{2i-1}) \\ &\leq \omega_s(\alpha_n) + \omega_t(\alpha_n) \\ &< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \\ \Rightarrow \alpha_n &< \alpha_n,\end{aligned}$$

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = p(Ax_{2i}, Bx_{2j+1})$ leads to contradiction by (1.i).

Case 3: The case where $\alpha_n = p(Ax_{2i}, Ax_{2j})$ leads to contradiction by (1.ii.2).

Case 4: If $\alpha_n = p(Bx_{2i+1}, Bx_{2j+1})$ then for k such that $2i+1 < 2k < 2j+1$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned}
\alpha_n &= p(Bx_{2i+1}, Bx_{2j+1}) \\
&\leq p(Ax_{2k}, Bx_{2i+1}) + p(Ax_{2k}, Bx_{2j+1}) \\
&\leq M_\omega(x_{2k}, x_{2i+1}) + M_\omega(x_{2k}, x_{2j+1}) \\
&\leq \ \omega_s(\alpha_n) + \omega_t(\alpha_n) \\
&< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \\
\Rightarrow \alpha_n &< \alpha_n,
\end{aligned}$$

which is a contradiction.

Case 5: If $\alpha_n = p(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \leq i, j \leq n-1$, then:

(5.i) If $Tx_{2i+1} = Ax_{2i}$, then we have

$$\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) = p(Ax_{2i}, Bx_{2j+1}),$$

which is a contradiction by (1.i).

(5.ii) Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence by Lemma 1.7 (i) we have

$$\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) \leq \max\{p(Bx_{2i-1}, Bx_{2j+1}), p(Ax_{2i}, Bx_{2j+1})\}.$$

This means we have either $p(Tx_{2i+1}, Bx_{2j+1}) \leq p(Bx_{2i-1}, Bx_{2j+1})$, which is a contradiction by Case 4 or $p(Tx_{2i+1}, Bx_{2j+1}) \leq p(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by (1.i).

Case 6: If $\alpha_n = p(Tx_{2i+1}, Ax_{2j})$ for some $0 \leq i, j \leq n-1$, then:

(6.i) If $Tx_{2i+1} = Ax_{2i}$, then we have $\alpha_n = p(Tx_{2i+1}, Ax_{2j}) = p(Ax_{2i}, Ax_{2j})$ which is not possible by (1.ii.2)

(6.ii) Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$, then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence $\alpha_n = p(Tx_{2i+1}, Ax_{2j}) \leq \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i}, Ax_{2j})\}$. This implies we have either $p(Tx_{2i+1}, Ax_{2j}) \leq p(Ax_{2j}, Bx_{2i-1})$, which is a contradiction by (1.i) or else we have $p(Tx_{2i+1}, Ax_{2j}) \leq p(Ax_{2i}, Ax_{2j})$, which is a contradiction by (1.ii.2).

Case 7: Suppose $\alpha_n = p(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \leq i, j \leq n-1$.

(7.i) If $Tx_{2j+1} = Ax_{2j}$, we have $\alpha_n = p(Tx_{2i+1}, Tx_{2j+1}) = p(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

(7.ii) Otherwise if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \text{seg}[Bx_{2j-1}, Ax_{2j}]$ and hence $\alpha_n = p(Tx_{2i+1}, Tx_{j+1}) \leq \max\{p(Tx_{2i+1}, Bx_{2j-1}), p(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $p(Tx_{2i+1}, Tx_{2j+1}) \leq p(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5 or else we have
$p(Tx_{2i+1}, Tx_{2j+1}) \leq p(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

Case 8: Let $\alpha_n = p(Sx_{2i}, Bx_{2j+1})$, for some $0 \leq i, j, \leq n-1$.

(8.i) If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$, then
$\alpha_n = p(Sx_{2i}, Bx_{2j+1}) = p(Bx_{2i-1}, Bx_{2j+1})$ which is not possible as per Case 4. Hence $i = 0$.

(8.ii) If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it means that
$Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$. This implies that
$\alpha_n = p(Sx_{2i}, Bx_{2j+1}) \leq \max\{p(Ax_{2i-2}, Bx_{2j+1}), p(Bx_{2i-1}, Bx_{2j+1})\}$.

This leads to a contradiction by (1.i) when
$p(Sx_{2i}, Bx_{2j+1}) \leq p(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens that $p(Sx_{2i}, Bx_{2j+1}) \leq p(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: Let us consider when $\alpha_n = p(Sx_{2i}, Sx_{2j})$, for some $0 \leq i < j \leq n-1$.

(9.i) If $i \geq 1$ and $Sx_{2j} = Bx_{2j-1}$ then we have
$\alpha_n = p(Sx_{2i}, Sx_{2j}) = p(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to Case 8. Hence $i = 0$.

(9.ii) If $i \geq 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it implies that $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$ and $p(Sx_{2i}, Sx_{2j}) \leq \max\{p(Sx_{2i}, Ax_{2j-2}), p(Sx_{2i}, Bx_{2j-1})\}$.

If it happens $p(Sx_{2i}, Sx_{2j}) \leq p(Sx_{2i}, Ax_{2j-2})$, we get a contradiction by Case 1. However if it happens that $p(Sx_{2i}, Sx_{2j}) \leq p(Sx_{2i}, Bx_{2j-1})$, then we get a contradiction by Case 8. Hence $i = 0$.

Case 10: Suppose $\alpha_n = p(Sx_{2i}, Tx_{2j+1})$, for some $0 \leq i, j \leq n-1$.

(10.i) If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$, then we have

$\alpha_n = p(Sx_{2i}, Tx_{2j+1}) = p(Tx_{2j+1}, Bx_{2i-1})$ which is not possible as per Case 8. Hence $i = 0$.

(10.ii) If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it implies that
$Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and
$p(Sx_{2i}, Tx_{2j+1}) \leq \max\{p(Tx_{2j+1}, Ax_{2i-2}), p(Tx_{2j+1}, Bx_{2i-1})\}$. This leads to contradictions by Case 6 and Case 5. Hence $i = 0$.

We have considered ten possible cases for α_n and conclude from Cases 1, 8, 9 and 10 that for some $0 \leq j \leq n-1$ we have

$$\alpha_n \in \{p(Sx_0, Sx_{2j}), p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1}), p(Sx_0, Tx_{2j+1})\}. \tag{2.6}$$

Note that, from the construction of sequence, $Sx_{2j} \in C$ implies
$Sx_{2j} = Bx_{2j-1}$. This leads to

$$p(Sx_0, Sx_{2j}) = p(Sx_0, Bx_{2j-1}). \tag{2.7}$$

For $Sx_{2j} \notin C$, we have $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$. From Lemma 1.7 (i), this implies

$$p(Sx_0, Sx_{2j}) \leq \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}. \tag{2.8}$$

From (2.7) and (2.8) we conclude that

$$p(Sx_0, Sx_{2j}) \leq \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}. \tag{2.9}$$

Using a similar argument we also have

$$p(Sx_0, Tx_{2j+1}) \leq \max\{p(Sx_0, Bx_{2j-1}), p(Sx_0, Ax_{2j})\}. \tag{2.10}$$

Applying (2.9) and (2.10) to (2.6) we get

$$\alpha_n \leq \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, 0 \leq j \leq n-1.$$

We have proved (2.3).

Consider the case where $\max\{p(Sx_0, Ax_{2j})\} \leq \max\{p(Sx_0, Bx_{2j+1})\}$, $0 \leq j \leq n-1$. Then, for some $u \in \{1, 2, \ldots, 5\}$, (2.3) implies

$$\begin{aligned} \alpha_n &\leq p(Sx_0, Bx_{2j+1}) \\ &\leq p(Sx_0, Ax_0) + p(Ax_0, Bx_{2j+1}), \text{ by (1.2)} \\ &\leq p(Sx_0, Ax_0) + \omega_u[\alpha_n)] \\ &\leq p(Sx_0, Ax_0) + 2\omega_u[\alpha_n)] \\ \Rightarrow \alpha_n - 2\omega_u[\alpha_n] &\leq p(Sx_0, Ax_0). \end{aligned}$$

Consider now, when $\max\{p(Sx_0, Ax_{2j})\} > \max\{p(Sx_0, Bx_{2j+1})\}$, $0 \leq j \leq n-1$. Then for some $v \in \{1, 2, \ldots, 5\}$, (2.3) implies

$$\begin{aligned} \alpha_n &\leq p(Sx_0, Ax_{2j}) \\ &\leq p(Sx_0, Ax_0) + p(Ax_0, Ax_{2j}), \text{ by (1.2)} \\ &\leq p(Sx_0, Ax_0) + 2\omega_v[\alpha_n)], \text{ by (1.ii.2)} \\ \Rightarrow \alpha_n - 2\omega_v[\alpha_n] &\leq p(Sx_0, Ax_0). \end{aligned}$$

Thus in both cases, we have for some $s \in \{1, 2, \ldots, 5\}$

$$\alpha_n - 2\omega_s[\alpha_n] \leq p(Sx_0, Ax_0). \tag{2.11}$$

By assumption (i) there is an $r_0 \in [0, +\infty)$ such that for each $s \in \{1, 2, \ldots, 5\}$, $r - 2\omega_s[r] > p(Sx_0, Ax_0)$ for $r > r_0$.

There is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n, we have

$$a_n - 2\omega_s[a_n] \leq p(Sx_0, Ax_0).$$

Thus by (2.11), $a_n \leq r_0$. Thus we have

$$a = \lim_{n \to +\infty} a_n = \text{diam}(D) \leq r_0.$$

We have hence proved that $\{Sx_{2n}\}, \{Tx_{2n+1}\}, \{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ are bounded sequences.

To prove that $\{Sx_{2n}\}, \{Tx_{2n+1}\}, \{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge in C, we reflect on the set

$$E_n = \left(\bigcup_{i=n}^{+\infty}\{Ax_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Sx_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Tx_{2i+1}\}\right), \tag{2.12}$$

$n = 2, 3, \ldots$

By (2.3) we have for $n = 2, 3, \ldots$

$$e_n := \text{diam}(E_n) \leq sup_{j \geq n}\{p(Sx_{2n}, Ax_{2j}), p(Sx_{2n}, Bx_{2j+1})\}. \tag{2.13}$$

If $Sx_{2n} = Bx_{2n-1}$ we have as in Case 1 and Case 8, for each $j \geq n$, and for some $u \in \{1, 2, \ldots, 5\}$

$$\begin{aligned} e_n &\leq sup_{j \geq n}\{p(Ax_{2j}, Bx_{2n-1}), p(Bx_{2j+1}, Bx_{2n-1})\}, n = 2, 3 \ldots \\ &\leq 2\omega_u[e_{n-1}]. \end{aligned} \tag{2.14}$$

If however $Sx_{2n} \neq Bx_{2n-1}$, it implies that $Sx_{2n} \in \text{seg}[Ax_{2n-2}, Bx_{2n-1}]$. Hence, as in Case 1 and Case 8, for each $j \geq n$ and for some
$v \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned} e_n \leq sup_{j \geq n}\{&p(Ax_{2n-2}, Ax_{2j}), p(Bx_{2n-1}, Ax_{2j}), p(Ax_{2n-2}, Bx_{2j+1}), \\ &p(Bx_{2n-1}, Bx_{2j+1})\}, n = 2, 3, \ldots \\ \leq\ & 2\omega_v[e_{n-2}]. \end{aligned} \tag{2.15}$$

By (2.14) and (2.15), there is a subsequence $\{\varepsilon_n\}$ of $\{e_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n

$$\begin{aligned} \varepsilon_n &\leq 2\omega_s[\varepsilon_{n-2}], n = 2, 3, \ldots \\ &< \varepsilon_{n-2}. \end{aligned} \tag{2.16}$$

We note that $e_n \geq e_{n+1}$ for every n. Let $\lim_{n \to +\infty} e_n = \lim_{n \to +\infty} \varepsilon_n = e$. We claim that $e = 0$.

If $e > 0$, then by (2.16) and assumption (i), we have

$$\begin{aligned} \lim_{n\to+\infty} \varepsilon_n &< \lim_{n\to+\infty} \varepsilon_{n-2} \\ &\Rightarrow e < e, \end{aligned}$$

which is a contradiction. Hence $e = 0$.

We recall from (2.13) that $e_n = Diam(E_n)$. Taking $n, m \to +\infty$ in (2.12), we get

$$\lim_{n,m\to+\infty} p(Ax_{2n}, Ax_{2m}) = \lim_{n,m\to+\infty} p(Bx_{2n+1}, Bx_{2m+1}) = 0. \tag{2.17}$$

This means both $\{A_{2n}\}$ and $\{B_{2n+1}\}$ are Cauchy sequences.

Because X is a complete partial metric space, this means there is $z \in X$ such that

$$\lim_{n\to+\infty} Ax_{2n} = \lim_{n\to+\infty} Bx_{2n+1} = z \text{ and } p(z, z) = 0. \tag{2.18}$$

Consider the subsequence Sx_{2n_k} of Sx_{2n} such that $Sx_{2n_k} = Bx_{2n_k-1}$. Taking $n_k \to +\infty$ we have

$$\lim_{n\to+\infty} Sx_{2n} = \lim_{n\to+\infty} Sx_{2n_k} = \lim_{n\to+\infty} Bx_{2n_k-1} = z, \text{ with } p(z, z) = 0. \tag{2.19}$$

Using a similar argument we have

$$\lim_{n\to+\infty} Tx_{2n+1} = z, \text{ with } p(z, z) = 0. \tag{2.20}$$

But SC, TC are both 0-complete. This implies $z \in SC$ and $z \in TC$.

As $z \in SC$, there is a point $u \in C$ such that $Su = z$. We show that u is a coincidence point of A, B and S.

$$\begin{aligned} p(Au, Bx_{2n+1}) \leq \max\{&\omega_1[p(Su, Tx_{2n+1})], \omega_2[p(Au, Su)], \\ &\omega_3[p(Bx_{2n+1}, Tx_{2n+1})], \omega_4[p(Au, Tx_{2n+1})], \\ &\omega_5[p(Su, Bx_{2n+1})]\} \\ = \max\{&\omega_1[p(z, Tx_{2n+1})], \omega_2[p(Au, z)], \omega_3[p(Bx_{2n+1}, Tx_{2n+1})], \\ &\omega_4[p(Au, Tx_{2n+1})], \omega_5[p(z, Bx_{2n+1})]\}. \end{aligned}$$

Taking $n \to +\infty$ and applying (2.19) and (2.20), we get

$$\begin{aligned} p(Au, z) &\leq \max\{\omega_1[p(z, z)], \omega_2[p(Au, z)], \omega_3[p(z, z], \omega_4[p(Au, z)], \omega_5[p(z, z)]\} \\ &\leq \omega_i[p(Au, z)] \text{ for some } i \in \{1, 2, \ldots, 5\} \\ &< p(Au, z) \text{ for } p(Au, z) > 0 \\ &\Rightarrow p(Au, z) = 0 \\ &\Rightarrow Au = z, \text{ from (1.1).} \end{aligned} \tag{2.21}$$

Using a similar procedure, when we expand $p(Ax_{2n}, Bu)$, we get $Bu = z$, making u a coincidence point of A, B and S. By the coincidental commutativity of S and A, we have $SAu = ASu \Rightarrow Sz = Az$.

In the same vein, $z \in TC$ means there is $v \in C$, such that $Tv = z$. We show that $Bv = z$.

$$\begin{aligned}
p(z, Bv) &= p(Au, Bv) \\
&\leq \max\{\omega_1[p(Su, Tv)], \omega_2[p(Su, Au)], \omega_3[p(Tv, Bv)], \\
&\quad \omega_4[p(Au, Tv)], \omega_5[p(Su, Bv)]\} \\
&= \max\{\omega_1[p(z, z)], \omega_2[p(z, z)], \omega_3[p(z, Bv)], \\
&\quad \omega_4[p(z, z)], \omega_5[p(z, Bv)]\} \\
&\leq \omega_j[p(z, Bv)] \text{ for some } j \in \{1, 2, \ldots, 5\} \\
&< p(z, Bv) \text{ for } p(z, Bv) > 0 \\
&\Rightarrow p(z, Bv) = 0 \\
&\Rightarrow Bv = z, \text{ from (1.1).}
\end{aligned}$$

Thus v is a coincidence point of B and T. By the coincidental commutativity property, we have

$$BTv = TBv \Rightarrow Bz = Tz.$$

Now consider the following:

$$\begin{aligned}
p(Az, Bz) &\leq \max\{\omega_1[p(Sz, Tz)], \omega_2[p(Sz, Az)], \omega_3[p(Tz, Bz)], \\
&\quad \omega_4[p(Az, Tz)], \omega_5[p(Sz, Bz)]\} \\
&= \max\{\omega_1[p(Az, Bz)], \omega_2[p(Az, Az)], \omega_3[p(Bz, Bz)], \\
&\quad \omega_4[p(Az, Bz)], \omega_5[p(Az, Bz)]\} \\
&\leq \omega_i[p(Az, Bz)] \text{ for } i \in \{1, 2, \ldots, 5\} \\
&< p(Az, Bz) \text{ for } p(Az, Bz) > 0 \\
\Rightarrow p(Az, Bz) &= 0 \\
\Rightarrow Az &= Bz.
\end{aligned}$$

Hence we have

$$Az = Bz = Sz = Tz. \tag{2.22}$$

Now we consider the following:

$$\begin{aligned} p(z, Bz) &= p(Au, Bz), \text{ from (2.21)} \\ &\leq \max\{\omega_1[p(Su, Tz)], \omega_2[p(Au, Su)], \omega_3[p(Bz, Tz)], \\ &\qquad \omega_4[p(Au, Tz)], \omega_5[p(Su, Bz)]\} \\ &= \max\{\omega_1[p(z, Bz)], \omega_2[p(z.z)], \omega_3[p(Bz, Bz)], \\ &\qquad \omega_4[p(z, Bz)], \omega_5[p(z, Bz)]\}. \end{aligned}$$

This implies

$$\begin{aligned} p(z, Bz) &\leq \omega_j[p(z, Bz)] \text{ for some } j \in \{1, 2, \ldots, 5\} \\ &< p(z, Bz) \text{ for } p(z, Bz) > 0 \\ \Rightarrow p(z, Bz) &= 0 \\ \Rightarrow Bz &= z \text{ by (1.1)}. \end{aligned}$$

From (2.22) we conclude that

$Az = Bz = Sz = Tz = z$,

meaning that z is a common fixed point of A, B, S and T.

We now show that z is unique. Suppose z' is also a common fixed point of A, B, S and T. We get

$$\begin{aligned} p(z, z') &= p(Az, Bz') \\ &\leq \max\{\omega_1[p(Sz, Tz')], \omega_2[p(Az, Sz)], \omega_3[p(Bz', Tz')], \\ &\qquad \omega_4[p(Az, Tz')], \omega_5[p(Sz, Bz')]\} \\ &= \max\{\omega_1[p(z, z')], \omega_2[p(z, z)], \omega_3[p(z', z')], \\ &\qquad \omega_4[p(z, z')], \omega_5[p(z, z')]\}. \end{aligned}$$

This implies

$$\begin{aligned} p(z, z') &\leq \omega_k[p(z, z'] \text{ for } k \in \{1, 2, \ldots, 5\} \\ &< p(z, z') \text{ for } p(z, z') > 0 \\ \Rightarrow p(z, z') &= 0 \\ \Rightarrow z &= z'. \end{aligned}$$

This proves that the common fixed point of A, B, S and T is unique. □

Remark 2.2. *Theorem 2.1 leads to corollaries if we set the following:*

(i) $A = B$;
(ii) $A = B, S = T$, *we get Theorem 1.9;*
(iii) $A = B, S = T = I$, *we get an extension of a theorem proved by Ćirić* [2] *into partial metric spaces;*
(iv $A = I$;
(v) $S = T$;
(vi) $x = y$.

The proof given works even when we define C as closed in (X, p^s). This lead to the following theorem.

Theorem 2.3. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a closed subset of X, the closure being taken with respect to (X, p^s). Let ∂C, the boundary of C in (X, p^s), be non-empty. Let $A, B, S, T : C \to X$. Let us assume that A, B, S and T satisfy the following conditions:*
(i) For every $x, y \in C, p(Ax, By) \leq M_\omega(x, y)$ where
$M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)], \omega_4[p(Ax, Ty)],$
$\omega_5[p(Sx, By)]\}, \omega_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3, 4, 5$, *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r/2$ for $r > 0$, and $\lim_{r\to+\infty}[r - 2\omega_i(r)] = +\infty$.*
(ii) $\partial C \subseteq SC, \partial C \subseteq TC$,
(iii) $Sx \in \partial C \Rightarrow Ax \in C; Tx \in \partial C \Rightarrow Bx \in C$,
(iv)$AC \cap C \subset TC, BC \cap C \subset SC$ and
(v) SC, TC are closed in C.
Then there exists a coincidence point $z \in C$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Here we give an example on the use Theorem 2.3, as it is better suited for the partial metric that we will use.

Example 2.4. *Consider the partial metric space $(\mathbb{R}_+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$. Let $C = [0, 2]$. We note that C is closed in the derived metric $p^s(x, y) = |x - y|$ and $\partial C = \{0, 2\}$.*

We define the mappings $A, B, S, T : C \to \mathbb{R}_+$ as follows:

$$Ax = \begin{cases} 3^x - 1, & x \in [0, 1] \\ 1, & x \in (1, 2] \end{cases}, \qquad Bx = \begin{cases} 4^x - 1, & x \in [0, 1] \\ 2, & x \in (1, 2] \end{cases},$$

$$Sx = \begin{cases} 27^x - 1, & x \in [0, 1] \\ 6, & x \in (1, 2] \end{cases}, \qquad Tx = \begin{cases} 64^x - 1, & x \in [0, 1] \\ 8, & x \in (1, 2] \end{cases}.$$

We have $AC = [0, 2]$ *and* $BC = [0, 3]$. *We also have* $TC = [0, 63]$ *and* $SC = [0, 26]$, *both of which are closed in* (X, p^s).

$Sx \in \partial C$ *implies* $z \in \{0, 1/3\} \subset C$. *Similarly* $Tx \in \partial C$ *implies* $x \in \{0, \ln 3/\ln 64\} \subset C$. *We also have* $\partial K = \{1, 3\} \subseteq SC, TC$.

We note that $\{S, A\}$ *and* $\{T, B\}$ *are both coincidentally commuting at* $x = 0$: *that is,* $SA(0) = AS(0)$ *and* $TB(0) = BT(0)$. *We also note that all four mappings are discontinuous at* 1.

Let us define the functions $g, h : C \to \mathbb{R}_+$ *as*

$$g(x) = \frac{27^x - 1}{4^x - 1}, \qquad h(x) = \frac{64^x - 1}{4^x - 1}.$$

Both g *and* h *are increasing functions. Using l'Hôpital rule, we can show that, as* $x \to 0$, *we have* $h(x) \to 3$ *and*

$$g(x) \to \frac{\log 27}{\log 4} = \frac{1}{0.42062}. \tag{2.23}$$

.

Hence, for $x \in [0, 1]$, *we have*

$$4^x - 1 \leq \frac{1}{3}(64^x - 1) \leq 0.43(64^x - 1). \tag{2.24}$$

We also have, from (2.23)

$$\begin{aligned} 4^x - 1 &\leq 0.42062(27^x - 1) \\ \Rightarrow 4^x - 1 &\leq 0.43(27^x - 1). \end{aligned} \tag{2.25}$$

When $x, y \in [0, 1]$, *with* $x \leq y$, *we have*

$$\begin{aligned} p(Ax, By) &= p(3^x - 1, 4^y - 1) \\ &= \max\{3^x - 1, 4^y - 1\} \\ &= 4^y - 1, \ \textit{because } x \leq y \\ &\leq 0.43(64^y - 1), \ \textit{by (2.24)} \\ &= 0.43Ty \\ &\leq 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

When $x, y \in [0,1]$, with $x > y$, we have

$$\begin{aligned} p(Ax, By) &= p(3^x - 1, 4^y - 1) \\ &= \max\{4^x - 1, 4^y - 1\}, \text{ because } 3^x - 1 \leq 4^x - 1 \\ &= 4^x - 1, \text{ because } x > y \\ &\leq 0.43(27^x - 1), \text{ by (2.25)} \\ &= 0.43Sx \\ &\leq 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

For $x, y \in (1,2]$ we have

$$\begin{aligned} p(Ax, By) &= p(1,2) \\ &= \max\{1,2\} \\ &= 2 \\ &< 0.43 \times 6 \\ &\leq 0.43 \max\{6, Ty\} \\ &= 0.43 \max\{Sx, Ty\}, \text{ because } Sx = 6 \\ &= 0.43p(Sx, Ty). \end{aligned}$$

Consider when $x \in [0,1], y \in (1,2]$ we get

$$\begin{aligned} p(Ax, By) &= p(3^x - 1, 2) \\ &= \max\{3^x - 1, 2\} \\ &= 2, \text{ because } 3^x - 1 \leq 2 \text{ for } x \in [0,1] \\ &< 0.43 \times 8 \\ &= 0.43Ty \text{ because } Ty = 8 \\ &\leq 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

Consider when $x \in (1,2], y \in (0,1)$.

$$\begin{aligned} p(Ax, By) &= p(1, 4^y - 1) \\ &= \max\{1, 4^y - 1\} \\ &= \begin{cases} 1 \\ 6, \quad y \in (0.5, 1] \end{cases} \end{aligned}$$

For $y \in [0, 0.5]$, we have

$$\begin{aligned} p(Ax, By) &= p(1, 4^y - 1) \\ &= 1 \\ &< 0.43 \times 6 \\ &= 0.43Sx \text{ because } Sx = 6 \\ &\leq 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

For $y \in (0.5, 1]$ we have

$$\begin{aligned} p(Ax, By) &= p(1, 4^y - 1) \\ &= 4^y - 1 \\ &\leq 0.43(64^y - 1), \text{ by (2.24)} \\ &= 0.43Ty \\ &\leq 0.43 \max\{Sx, Ty\} \\ &= 0.43p(Sx, Ty). \end{aligned}$$

Thus in all cases, for every $x, y \in C$ we have

$$p(Ax, By) \leq 0.43p(Sx, Ty) \leq M_\omega(x, y),$$

where

$$\begin{aligned} M_\omega(x, y) = \max \big\{&\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(Ay, Ty)], \\ &\omega_4[p(Ax, Ty)], \omega_5[p(Sx, Ay)]\big\}, \end{aligned}$$

with $\omega_i[r] = 0.43r$. We note that $\omega_i[r] = 0.43r < \frac{1}{2}r$.
As $r - 2\omega_i[r] = r - 2 \times 0.43r = 0.14r$, we also have
$\lim_{r \to +\infty} r - 2\omega_i[r] = \lim_{r \to +\infty} 0.14r = +\infty$.

Thus all the conditions of Theorem 2.3 are satisfied and 0 is the unique common fixed point of A, B, S and T.

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Authors' addresses:
Terentius Rugumisa, Faculty of Science, Technology and Environmental Studies, The Open University of Tanzania, Tanzania. e-mail: `rterentius@gmail.com`
Santosh Kumar, Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Tanzania. e-mail: `drsengar2002@gmail.com`
Mohammad Imdad, Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India. e-mail: `mhimdad@gmail.com`