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Nonstandard Discretizations of the SIS Epidemiological Model with and without Diffusion

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This work is dedicated to Professor Ronald E. Mickens, founder of the nonstandard finite difference method, on the occasion of his 70^{th} birthday.

Abstract. We design and investigate the reliability of various nonstandard finite difference (NSFD) schemes for SIS-type epidemiological models. The success of the study is due to an innovative use of Mickens' rules of complex denominator functions and nonlocal approximation of nonlinear terms. For the classical SIS-ODE model, we construct for the first time a nonstandard Runge-Kutta method, which is shown to be of order four. We also consider two new NSFD schemes which faithfully replicate the property of the continuous model of having the value $\mathcal{R}_0 = 1$ of the basic reproduction parameter as a forward bifurcation: the disease-free equilibrium is globally asymptotically stable when $\mathcal{R}_0 < 1$; it is unstable when $\mathcal{R}_0 > 1$ and there appears a unique locally asymptotically stable endemic equilibrium in this case. The latter schemes are further used to derive NSFD schemes that are dynamically consistent with the positivity and boundedness properties of the SIS-diffusion model for the spatial spread of disease. Numerical simulations that support the theory are provided.

1. Introduction

The SIS epidemiological model is the simplest representation for a disease in which recovery does not guarantee immunity. The population N is divided into two disjoint compartments: S, the susceptible individuals and I, the infective individuals. A susceptible individual that becomes infective returns to the class of susceptible individuals after recovery.

Though being simple, the SIS model has been studied extensively and is still receiving a lot of attention in order to have an insight on more complex models or to develop them [**5**, **6**, **12**, **17**].

The study of the SIS model from the numerical analysis point of view, which is the focus of this work, is even more relevant because most models of interest cannot be solved completely by analytical techniques. The specific SIS models we considered involve vital dynamics with either a diffusion term in the case when the

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disease spreads in space or without this term. Theoretical study of diseases in which recovery does not guarantee immunity and which spread spatially in a diffusion manner can be found in [**15**, **20**, **21**]. For further investigations on reaction-diffusion models in epidemiology, we can mention [**1**, **2**] for the SIS case and [**10**, **20**, **22**] for more general situations.

There is a double great deal in our study. On the one hand, we design a higher order nonstandard finite difference scheme, an approach which to the authors' best knowledge has not been considered in the literature. On the other hand, our NSFD schemes replicate the essential dynamics of the continuous models.

One of the main strategies towards the power of our schemes in being dynamically consistent is that Mickens' rule on the nonlocal approximation of nonlinear terms is implemented in an innovative manner. At the point $(x,t) = (n\Delta x, k\Delta t)$, the approximations

$$\frac{SI}{N}|_{(x,t)} \simeq \begin{cases} \dfrac{S_n^{k+1} I_n^k}{S_n^{k+1} + I_n^k} \\ \\ \dfrac{S_n^{k+1}(I_{n+1}^k + I_n^k + I_{n-1}^k)}{3N_n^{k+1}} \end{cases} \tag{1.1}$$

are used for the nonlinear term $\frac{SI}{N}$ that occurs in the SIS model instead of the local discretization $\frac{S_n^k I_n^k}{N_n^k}$. Though both resulting NSFD schemes preserve the positivity and the boundedness of the solutions, the scheme that is based on the second approximation in (1.1) does this in a more precise manner like the continuous model [**15**, **21**]: $0 \leq I_n^k,\ S_n^k \leq N_n^k \leq K \Rightarrow 0 \leq I_n^{k+1},\ S_n^{k+1} \leq N_n^{k+1} \leq K$. The scheme that is based on the first approximation in (1.1) is further dynamically consistent with the global asymptotic stability of the disease-free equilibrium and the local asymptotic stability of the endemic equilibrium.

The rest of the work is organized as follows: in the next section, we recall the SIS model (with and without diffusion) and summarize its main qualitative properties. Section 3 is devoted to the presentation of the nonstandard Runge-Kutta method as a higher order method. Schemes for the SIS model are investigated in Section 4 where the emphasis is on the preservation of the dynamics of the continuous model. The material presented in Section 4 is exploited in Section 5 to develop NSFD schemes that are dynamically consistent with the SIS-diffusion model. Concluding remarks as to how our work fits in the literature are reported in Section 6. Throughout the relevant sections, the work is enriched with numerical tests that support the theory.

2. Qualitative properties of the model

We assume that a population of size N consists of only two disjoint compartments, susceptible S and infective I, which interact:

$$S + I = N. \tag{2.1}$$

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The SIS model with standard incidence and without disease induced death rate reads as follows [**6**, **12**]:

$$S_t = \mu K - \frac{\beta IS}{N} - \mu S + \gamma I \tag{2.2}$$

$$I_t = \frac{\beta IS}{N} - (\mu + \gamma) I. \tag{2.3}$$

Here the constant $\mu > 0$ is the birth or natural death rate, the constant $\gamma > 0$ is the rate infective individuals return into the susceptible class, β is the contact number and the constant K is the carrying capacity of the environment.

By adding (2.2) and (2.3), we get the conservation law

$$N_t = \mu(K - N). \tag{2.4}$$

With

$$\mathcal{R}_0 = \frac{\beta}{\mu + \gamma} \tag{2.5}$$

representing the basic reproduction number, the SIS model is equivalent to the scalar equation

$$I_t = \beta(1 - \frac{1}{\mathcal{R}_0}) \left(1 - \frac{I}{N(1 - \frac{1}{\mathcal{R}_0})}\right) I. \tag{2.6}$$

The disease-free equilibrium of the SIS model is $E_0 = (\bar{S}, \bar{I}) = (K, 0)$ while the endemic equilibrium is $E_\infty = (S_\infty, I_\infty) = \left(\frac{K}{\mathcal{R}_0}, K(1 - \frac{1}{\mathcal{R}_0})\right)$ whenever $\mathcal{R}_0 > 1$.

The qualitative properties given in the next theorem are well-known [**12**]:

THEOREM 2.1. (1) *The SIS model (2.2)-(2.3) is a dynamical system on the biologically feasible region*

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : \ 0 \leq S + I = N \leq K\}.$$

(2) *If $\mathcal{R}_0 < 1$, the disease–free equilibrium $E_0 = (K, 0)$ is globally asymptotically stable.*

(3) *If $\mathcal{R}_0 > 1$, E_0 is unstable and the endemic equilibrium $E_\infty = (S_\infty, I_\infty)$ is locally asymptotically stable.*

The next step is to model the dynamics of $I(x, t)$ and $S(x, t)$ by simple diffusion (for concrete situations see [**20**, **21**]). Our additional assumption is that the dispersion is completely random and has the same structural properties. For the spread of the disease in space, we consider the following model with standard incidence:

$$S_t = \mu K - \frac{\beta IS}{N} - \mu S + \gamma I + S_{xx} \tag{2.7}$$

$$I_t = \frac{\beta IS}{N} - (\mu + \gamma) I + I_{xx}. \tag{2.8}$$

More general expressions of the SIS-reaction diffusion system (2.7)-(2.8) are available in the literature. See for instance [**1**, **2**, **20**] where contact and recovery rates depend on the space variable, while the diffusion coefficients are different. In this work where the focus is on numerical discretizations, we adopt two simplifications. Firstly, we assume that the diffusion coefficients are the same for both classes.

By adding (2.7) and (2.8), we have the conservation law:

$$N_t = \mu(K - N) + N_{xx}. \tag{2.9}$$

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Secondly, given our interest in the spread of the disease in space, we assume that $\mathcal{R}_0 > 1$. Thus the reaction-diffusion system (2.7)-(2.8) is equivalent to a scalar reaction-diffusion equation in the dependent variable I:

$$I_t = \beta(1 - \frac{1}{\mathcal{R}_0}) \left(1 - \frac{I}{N(1 - \frac{1}{\mathcal{R}_0})}\right) I + I_{xx}. \tag{2.10}$$

In practice, we first solve Equation (2.9) and use the solution to solve Equation (2.10). This is done and guaranteed according to the following known result [**15**, **22**, **24**]:

THEOREM 2.2. *Assume that $N_0 : \mathbb{R} \to \mathbb{R}$ and $I_0 : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $0 \leq I_0(x) \leq N_0(x) \leq K$. Then, Equations (2.9)-(2.10) admit unique solutions $N : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ and $I : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ that satisfy the condition $0 \leq I(x, t) \leq N(x, t) \leq K$ and the initial conditions*

$$N(x, 0) = N_0(x); \quad I(x, 0) = I_0(x). \tag{2.11}$$

The next theorem for the qualitative analysis of equilibrium points follows [**22**], as discussed in [**24**] by linearization and the properties associated with the Sturm-Liouville problem.

THEOREM 2.3. *The equilibrium solution $\bar{N} = K$ of the conservation law (2.9) is globally asymptotically stable. For $\mathcal{R}_0 > 1$, the equilibrium solution $\bar{I}_0 = 0$ of (2.10) is unstable whereas the equilibrium solution $\bar{I}_\infty = K(1 - \frac{1}{\mathcal{R}_0})$ is locally asymptotically stable.*

REMARK 2.4. For the extended SIS-diffusion reaction model mentioned above, the stability of the equilibria, including the global asymptotic stability of the disease-free equilibrium, are investigated in [**1**] in terms of the basic reproduction number defined there.

Often, the disease is spread in space as a wave with speed c. It is therefore legitimate to seek for traveling wave solutions in the form

$$I(x, t) = W(z), \; z = x - ct \in \mathbb{R}$$

such that

$$\lim_{z \to \infty} W(z) = 0 \text{ and } \lim_{z \to -\infty} W(z) = K(1 - \frac{1}{\mathcal{R}_0}).$$

We have the following result [**15**, **20**, **22**].

THEOREM 2.5. *A traveling wave solution to (2.9)-(2.10) exists if and only if $N = K = constant$ and $c \geq 2\sqrt{\beta(1 - \frac{1}{\mathcal{R}_0})}$.*

3. Nonstandard Runge-Kutta Method for the SIS Model

The classical explicit k-stage Runge-Kutta method for determining discrete solutions (x_n) at the time $t_n = nh$ for the solution of the initial value problem

$$x_t = f(x), \; x(0) = x_0 \tag{3.1}$$

is defined by ([14])

$$x_{n+1} = x_n + h \sum_{i=1}^{k} b_i K_i, \tag{3.2}$$

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where the discrete derivatives K_i evaluated at the intermediate times $t_n + c_i h$ are given by

$$K_i = f\left(x_n + h\sum_{j=1}^{i} a_{ij}K_j\right),\ i = 1, 2, ..., k;\ c_i = \sum_{j=1}^{i} a_{ij}, \tag{3.3}$$

$(a_{ij})_{1\le i,\, j\le k}$ is a lower triangular matrix with zero diagonal and the weights (b_i) satisfy the consistency condition

$$\sum_{i=1}^{k} b_i = 1. \tag{3.4}$$

We restrict our study to $1 \le k \le 4$ because for this case it is well-known that the explicit k-stage Runge-Kutta method is of order k. More precisely, the local truncation error T_{n+1} is given by

$$T_{n+1} := x(t_{n+1}) - x(t_n) - h\sum_{i=1}^{k} b_i K_i = \mathcal{O}(h^{k+1}), \tag{3.5}$$

where the exact solution $x(t_n)$ is used in the definition of K_i given in (3.3).

On the other hand, we assume that (3.1) has a finite number of equilibrium points $\bar{x}$ which are all hyperbolic. Consider the linearized differential equation

$$z_t = Jz,\ z := x - \bar{x}, \tag{3.6}$$

where J, the Jacobian matrix of f at each equilibrium point $\bar{x}$ of (3.1), is supposed to be a diagonalizable $m \times m$ matrix with eigenvalues λ_l:

$$Q^{-1}JQ = \Lambda := diag(\lambda_1, \lambda_2, ..., \lambda_m). \tag{3.7}$$

For x_0 near each equilibrium point $\bar{x}$, the solution $x(t)$ of (3.1) behaves like the solution

$$z(t) = e^{tJ}z_0, \tag{3.8}$$

of (3.6) and has the asymptotic behavior

$$\lim_{t\to\infty} \|z(t)\| = 0 \tag{3.9}$$

if and only if all the eigenvalues λ_l have negative real parts.

It can be shown that the Runge-Kutta method applied to (3.6) reads

$$z_{n+1} = R(hJ)z_n, \tag{3.10}$$

where [**14**]

$$R(hJ) := Q\ diag[R(\lambda_l h)]Q^{-1} \text{ and } R(\lambda_l h) = \sum_{j=0}^{k} \frac{1}{j!}(\lambda_l h)^j. \tag{3.11}$$

Hence, (z_n) replicates the property (3.9) or $z_n \to 0$ as $n \to \infty$ if and only if

$$|R(\lambda_l h)| < 1\ \forall l, \tag{3.12}$$

which is known as the property of absolute stability of the Runge-Kutta method for each value $\lambda_l h$.

The necessity of modifying the k-stage Runge-Kutta method comes here. Consider a function $\varphi : \mathbb{R} \to \mathbb{R}_+$ such that $0 < \varphi(z) < 1$ for $z > 0$ and

$$\varphi(z) = z + \mathcal{O}(z^{k+1}). \tag{3.13}$$

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For instance, we can take

$$\varphi(z) = \frac{z}{1 + c\, z^k}, \tag{3.14}$$

where c is a suitable positive constant.

We consider the function

$$\phi(h) = \frac{\varphi(qh)}{q}, \tag{3.15}$$

where $q \geq |\lambda|$ and λ traces all the eigenvalues of all the Jacobian matrices at the equilibrium points $\bar{x}$ for (3.1). Our Nonstandard (NS) Runge-Kutta method is given by

$$x_{n+1} = x_n + \phi(h) \sum_{i=1}^{k} b_i K_i, \tag{3.16}$$

where all the K_i's are defined as in (3.3) but with $\phi(h)$ in place of h.

By construction, we have the following result:

THEOREM 3.1. *For $1 \leq k \leq 4$, the NS Runge-Kutta method (3.16) is of order k. Furthermore, any equilibrium point $\bar{x}$ of (3.1) is a fixed-point of the scheme (3.16) in such a way that the stability property of $\bar{x}$ is preserved. More precisely, for any $h > 0$ and for $\|x - \bar{x}\|$ small, the discrete solution (x_n) given by (3.16) satisfies $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$ if $Re\lambda_l < 0$ for all λ_l whereas $\|x_n - \bar{x}\| \to \infty$ if there exists at least one eigenvalue λ_l with $Re\lambda_l > 0$.*

REMARK 3.2. The NS Runge-Kutta scheme (3.16) is not elementary stable in the sense of [**4**] because it can have spurious fixed-points.

We investigate the performance of the NS Runge-Kutta method (3.16) on the SIS model in (2.2)-(2.3) or (2.6). We start with the case when the total population is constant: $N = K$. Then, the system (2.2)-(2.3) is the logistic equation

$$I_t = \beta(1 - \frac{1}{\mathcal{R}_0}) \left(1 - \frac{I}{K(1 - \frac{1}{\mathcal{R}_0})}\right) I =: f(I), \tag{3.17}$$

the exact solution of which is known to be

$$I(t) = \frac{K(1 - \frac{1}{\mathcal{R}_0})I_0}{I_0 + \left(K(1 - \frac{1}{\mathcal{R}_0}) - I_0\right) e^{-\beta(1-\frac{1}{\mathcal{R}_0})t}}. \tag{3.18}$$

To be more specific, we consider the nonstandard analogue of the classical fourth order Runge-Kutta method ($k = 4$). In this case, the choice (3.15)-(3.16) yields:

$$\phi(h) = \frac{h}{1 + c[|\beta(1 - \frac{1}{\mathcal{R}_0})|h]^4}. \tag{3.19}$$

The nonstandard Runge-Kutta scheme is

$$\frac{I_{n+1} - I_n}{\phi(h)} = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \tag{3.20}$$

where

$$l_1 = f(I_n), \quad l_2 = f\left(I_n + \frac{1}{2}\phi(h)l_1\right), \quad l_3 = f\left(I_n + \frac{1}{2}\phi(h)l_2\right) \quad l_4 = f\left(I_n + \phi(h)l_3\right).$$

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The second case of the interest is when N is not constant. Then (2.2)-(2.4) is equivalent to

$$I_t = \beta(1-\frac{1}{\mathcal{R}_0})\left(1-\frac{I}{N(1-\frac{1}{\mathcal{R}_0})}\right)I =: f(I,N), \tag{3.21}$$

$$N_t = \mu(K-N). \tag{3.22}$$

The eigenvalues of the Jacobian matrices at the equilibrium points being $\beta(\frac{1}{\mathcal{R}_0}-1)$ and $-\mu$, we take $q = |\beta(\frac{1}{\mathcal{R}_0}-1)|+\mu$ so that (3.14) and (3.15) yield

$$\phi(h) = \frac{h}{1+c[(|\beta(\frac{1}{\mathcal{R}_0}-1)|+\mu)h]^4}. \tag{3.23}$$

Then the Nonstandard Runge-Kutta scheme takes the form

$$\frac{N_{n+1}-N_n}{\phi(h)} = \frac{1}{6}(k_1+2k_2+2k_3+k_4), \quad \frac{I_{n+1}-I_n}{\phi(h)} = \frac{1}{6}(l_1+2l_2+2l_3+l_4), \tag{3.24}$$

where

$$\begin{aligned} k_1 &= \mu(K-N_n), & l_1 &= f(I_n,N_n)\\ k_2 &= \mu\left(K-N_n-\tfrac{1}{2}\phi(h)k_1\right), & l_2 &= f\left(I_n+\tfrac{1}{2}\phi(h)l_1, N_n+\tfrac{1}{2}\phi(h)k_1\right)\\ k_3 &= \mu\left(K-N_n-\tfrac{1}{2}\phi(h)k_2\right), & l_3 &= f\left(I_n+\tfrac{1}{2}\phi(h)l_2, N_n+\tfrac{1}{2}\phi(h)k_2\right)\\ k_4 &= \mu\left(K-N_n-\phi(h)k_3\right), & l_4 &= f\left(I_n+\phi(h)l_3, N_n+\phi(h)k_3\right). \end{aligned}$$

The discrete solutions for $I(t)$ given by (3.20) are presented in Table 1 and Table 2 for different values of h. The results show that reducing the step size h by half results in reduction of the error by a factor of sixteen, showing a fourth order of convergence. This is in agreement with Theorem 3.1. The excellent performance of the nonstandard Runge-Kutta method is further illustrated in Fig 1, which shows the global asymptotic stability (GAS) of the disease-free equilibrium (DFE) ($\mathcal{R}_0 < 1$) as well as in Fig 2 which displays the local asymptotic stability (LAS) of the endemic equilibrium (EE) and the instability of the DFE ($\mathcal{R}_0 > 1$). Note also that all discrete solutions are positive as should be, whereas the standard Runge-Kutta method could produce negative solutions as shown in Fig 5.

We now illustrate the performance of the NS Runge-Kutta method (3.24) in the case when N is not constant. The discrete solution for $I(t)$ is shown in Fig 3, which supports the existence of a locally asymptotically stable EE and the instability of the DFE when $\mathcal{R}_0 > 1$.

In the case when N is not constant, the exact solution of (3.22) is known. Since this equation is decoupled from the other, we can plug its solution in (3.21) and then use the Runge-Kutta method for this scalar equation. It should also be noted that the solution N of (3.22) tends to K as $t \to \infty$. Thus Equation (3.21) is asymptotically the same as Equation (3.17). This fact is apparent on comparing the profiles of the solutions in Fig 2 and Fig 3 when $\mathcal{R}_0 > 1$. For $\mathcal{R}_0 < 1$, the profile of the solutions are given in Fig 1 and Fig 4.

For comparison, the standard Runge-Kutta method is shown in Fig 5 and Fig 6. It is apparent that this method is not dynamically consistent with positivity of solutions and the property of the endemic equilibrium point.

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TABLE 1. NS Runge-Kutta method with $c = 10$, $K = 200$, $\mathcal{R}_0 = 0.75$

Time t	*Exact scheme*	NS-RK *solution*	*Error* : $h = 0.5$ $\phi = 0.2487$	NS-RK *solution*	*Error* : $h = 0.25$ $\phi = 0.4621$
$0.0000e+00$	$8.0000e+01$	$8.0000e+01$	$0.0000e+00$	$8.0000e+01$	$0.0000e+00$
$2.0000e+01$	$5.3135e+00$	$5.3142e+00$	$-7.2150e-04$	$5.3136e+00$	$-4.5089e-05$
$4.0000e+01$	$6.7274e-01$	$6.7291e-01$	$-1.7050e-04$	$6.7275e-01$	$-1.0655e-05$
$6.0000e+01$	$9.0258e-02$	$9.0292e-02$	$-3.3987e-05$	$9.0261e-02$	$-2.1239e-06$
$8.0000e+01$	$1.2201e-02$	$1.2207e-02$	$-6.1162e-06$	$1.2201e-02$	$-3.8218e-07$
$1.0000e+02$	$1.6509e-03$	$1.6520e-03$	$-1.0341e-06$	$1.6510e-03$	$-6.4616e-08$

TABLE 2. NS Runge-Kutta method with $c = 2$, $\mathcal{R}_0 = 3.25$

Time t	*Exact scheme*	NS-RK *solution*	*Error* : $h = 0.5$	NS-RK *solution*	*Error* : $h = 0.25$
$0.0000e+00$	$8.0000e+01$	$8.0000e+01$	$0.0000e+00$	$8.0000e+01$	$0.0000e+00$
$4.0000e+00$	$1.0556e+02$	$1.0514e+02$	$4.1069e-01$	$1.0553e+02$	$2.5775e-02$
$8.0000e+00$	$1.1033e+02$	$1.1020e+02$	$1.2655e-01$	$1.1032e+02$	$7.6427e-03$
$1.2000e+01$	$1.1100e+02$	$1.1098e+02$	$2.7063e-02$	$1.1100e+02$	$1.5747e-03$
$1.6000e+01$	$1.1110e+02$	$1.1109e+02$	$5.0887e-03$	$1.1110e+02$	$2.8534e-04$
$2.0000e+01$	$1.1111e+02$	$1.1111e+02$	$8.9602e-04$	$1.1111e+02$	$4.8401e-05$

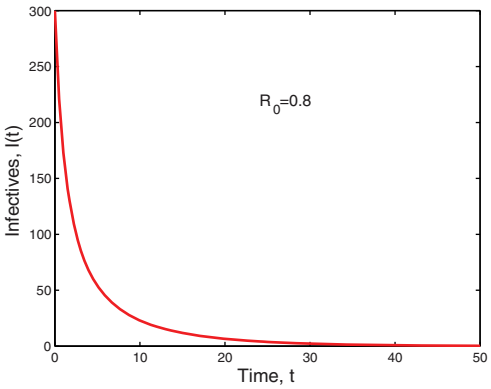


FIGURE 1. NS Runge-Kutta scheme (3.20) with $I_0 = 300$ for $\mathcal{R}_0 < 1$.

4. NSFD scheme for the SIS model

In preparation of the next section, we look at two alternative NSFD schemes for the SIS model (2.2)-(2.4). They are motivated by the exact scheme of the logistic

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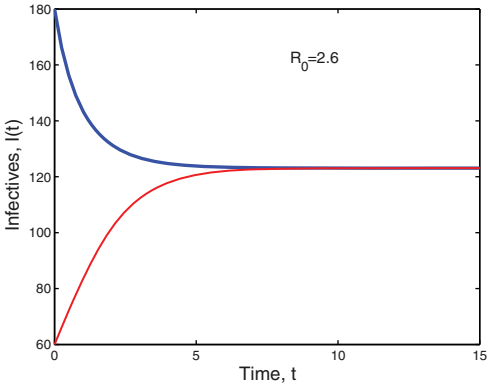


FIGURE 2. NS Runge-Kutta scheme (3.20) with $I_0 = 60$, $I_0 = 180$ for $\mathcal{R}_0 > 1$.

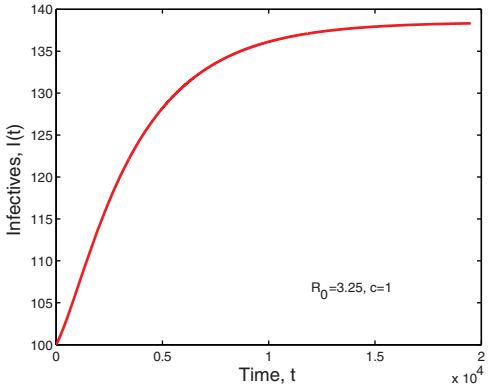


FIGURE 3. NS Runge-Kutta scheme for (3.21) with $K = 200$ for $\mathcal{R}_0 > 1$.

equation (3.17) that reads

$$\frac{I^{k+1} - I^k}{\phi} = \beta(1 - \frac{1}{\mathcal{R}_0}) \left(1 - \frac{I^{k+1}}{K(1 - \frac{1}{\mathcal{R}_0})}\right) I^k, \tag{4.1}$$

with the denominator function $\phi \equiv \phi(\Delta t) = \frac{1-exp[-|\beta(1-\frac{1}{\mathcal{R}_0})|\Delta t]}{|\beta(1-\frac{1}{\mathcal{R}_0})|}$ satisfying the asymptotic relation

$$\phi(\Delta t) = \Delta t + \mathcal{O}((\Delta t)^2). \tag{4.2}$$

In view of our future need, we however choose the following denominator function which satisfies (4.2):

$$\phi = \frac{1 - e^{-(\mu+\gamma)\Delta t}}{\mu + \gamma}. \tag{4.3}$$

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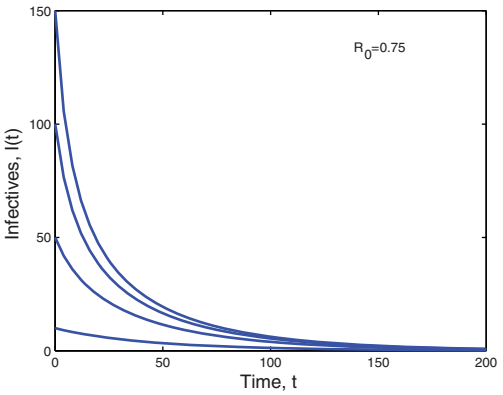


FIGURE 4. NS Runge-Kutta scheme for (3.21) with $c = 2$ and $K = 200$ for $\mathcal{R}_0 < 1$.

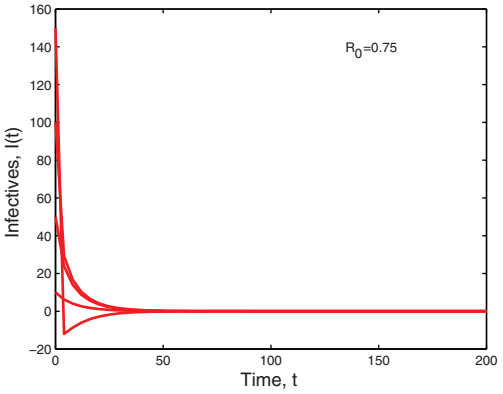


FIGURE 5. Standard Runge-Kutta scheme for (3.21) with $K = 200$ for $\mathcal{R}_0 < 1$.

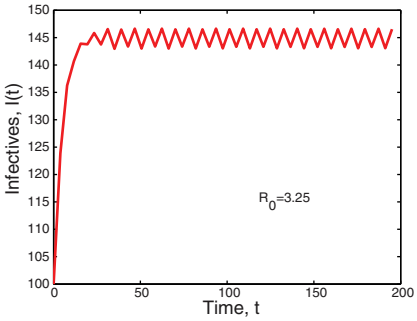


FIGURE 6. Standard Runge-Kutta scheme for (3.21) with $K = 200$ for $\mathcal{R}_0 > 1$.

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The first NSFD scheme is

$$\frac{S^{k+1}-S^k}{\phi} = \mu K - \frac{\beta S^{k+1} I^k}{S^{k+1}+I^k} - \mu S^{k+1} + \gamma I^k \tag{4.4}$$

$$\frac{I^{k+1}-I^k}{\phi} = \frac{\beta S^{k+1} I^k}{S^{k+1}+I^k} - \mu I^{k+1} - \gamma I^k. \tag{4.5}$$

It leads to the conservation law

$$\frac{N^{k+1}-N^k}{\phi} = \mu(K-N^{k+1}) \text{ or } N^{k+1} = \frac{\mu\phi K + N^k}{1+\mu\phi}. \tag{4.6}$$

The second NSFD scheme is based on the following explicit conservation law:

$$\frac{N^{k+1}-N^k}{\phi} = \mu(K-N^k) \text{ or } N^{k+1} = \mu\phi K + (1-\mu\phi)N^k. \tag{4.7}$$

Once N^{k+1} is obtained from (4.7), S^{k+1} and I^{k+1} are obtained as follows:

$$\frac{S^{k+1}-S^k}{\phi} = \mu K - \frac{\beta S^{k+1} I^k}{N^{k+1}} - \mu S^k + \gamma I^k \tag{4.8}$$

$$\frac{I^{k+1}-I^k}{\phi} = \frac{\beta S^{k+1} I^k}{N^{k+1}} - (\mu+\gamma) I^k. \tag{4.9}$$

By using

$$S^{k+1} + I^{k+1} = N^{k+1}, \tag{4.10}$$

the system (4.8)-(4.9) is equivalent to the scheme

$$\frac{I^{k+1}-I^k}{\phi} = \beta(1-\frac{1}{\mathcal{R}_0})\left(1 - \frac{I^{k+1}}{N^{k+1}(1-\frac{1}{\mathcal{R}_0})}\right) I^k, \tag{4.11}$$

which, as indicated earlier, is inspired by the exact scheme of the logistic equation when N is constant.

The implementation of the second NSFD scheme is straightforward because (4.8)-(4.9) or equivalently (4.11) can be written as

$$S^{k+1} = \frac{\mu\phi(K-S^k) + S^k + \gamma\phi I^k}{1+\frac{\beta\mu\phi I^k}{N^{k+1}}} \tag{4.12}$$

$$I^{k+1} = \left(\frac{\beta\phi S^{k+1}}{N^{k+1}} + 1 - (\mu+\gamma)\phi\right) I^k. \tag{4.13}$$

On the contrary, more attention is needed for the implementation of the first NSFD scheme (4.4)-(4.5). This is achieved by using the Gauss-Seidel cycle as described below.

Assume that $S^k \geq 0$ and $I^k \geq 0$ are known and put

$$\begin{aligned} A &= 1+\mu\phi \\ B &= (1+(\beta+\mu-\gamma)\phi)\, I^k - (S^k + \mu\phi K) \\ C &= -\left(S^k + \mu\phi K + \gamma\phi I^k\right) I^k. \end{aligned}$$

Then Equation (4.4) is equivalent to the following quadratic equation in S^{k+1}:

$$A(S^{k+1})^2 + BS^{k+1} + C = 0. \tag{4.14}$$

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The unique positive root of (4.14) is given by

$$S^{k+1} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \tag{4.15}$$

whereas (4.5) yields:

$$I^{k+1} = \frac{\left(\frac{\beta}{S^{k+1}+I^k}\phi S^{k+1} + (1-\gamma\phi)\right) I^k}{1+\mu\phi}. \tag{4.16}$$

Proceeding as in [**9**, **24**] and observing from (4.3) that

$$1 - (\gamma + \mu)\phi \geq 0, \tag{4.17}$$

we have the following result:

THEOREM 4.1. *The NSFD scheme (4.4)-(4.5) is dynamically consistent with the properties of the SIS model stated in Theorem 2.1. More precisely:*

(1) *The NSFD scheme is a dynamical system on*

$$\Omega = \{(S, I) \in \mathbb{R}^2_+ : 0 \leq S + I = N \leq K\}.$$

(2) *If $\mathcal{R}_0 < 1$, the disease-free fixed point $E_0 = (K, 0)$ is globally asymptotically stable. If $\mathcal{R}_0 > 1$, E_0 is unstable; $E_\infty = (S_\infty, I_\infty)$ is the unique endemic fixed point and it is locally asymptotic stable.*

The second NSFD scheme will be essentially used for the diffusion problem. For this reason, we assume that the basic reproduction number is greater than 1.

THEOREM 4.2. *The NSFD scheme (4.7) is dynamically consistent with respect to positivity, boundedness and monotonicity of the exact solution of the conservation law as well as with the GAS of the equilibrium solution $\bar{N} = K$. The NSFD scheme (4.8)-(4.9) or (4.11) preserves the positivity and boundedness properties in the following specific manner:*

$$0 \leq I^k \leq N^k \leq K \Rightarrow 0 \leq I^{k+1} \leq N^{k+1} \leq K.$$

PROOF. Assume that $0 \leq I^k \leq N^k \leq K$. Then from Equation (4.7), we have $0 \leq N^{k+1} \leq K$ and

$$K - N^{k+1} = (1 - \mu\phi)(K - N^k) \leq K - N^k$$

from which we deduce that $N^k \leq N^{k+1}$ and

$$K - N^{k+1} = (1 - \mu\phi)^{k+1}(K - N^0) \to 0 \text{ as } k \to \infty, \text{ by (4.17)}.$$

On the other hand (4.11) is equivalent to

$$I^{k+1} = \frac{N^{k+1}(1 - \frac{1}{\mathcal{R}_0})\left(\beta(1 - \frac{1}{\mathcal{R}_0})\phi + 1\right) I^k}{N^{k+1}(1 - \frac{1}{\mathcal{R}_0}) + \beta(1 - \frac{1}{\mathcal{R}_0})\phi I^k}, \tag{4.18}$$

which shows that $I^{k+1} \geq 0$ because $\mathcal{R}_0 > 1$.

To prove the remaining part of the theorem, we assume that $0 \leq S^k \leq N^k$. By using (4.10) and subtracting Equation (4.18) from Equation (4.7), we have $S^{k+1} \geq 0$. Thus, it follows from (4.10) that $0 \leq I^{k+1} \leq N^{k+1} \leq K$. □

THEOREM 4.3. *Under the condition $\mathcal{R}_0 > 1$, the NSFD scheme (4.11) or (4.8)-(4.9) is elementary stable.*

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PROOF. It is clear that the equilibrium points $\bar{I} = 0$ and $\bar{I} = K(1 - \frac{1}{\mathcal{R}_0})$ of the continuous model (3.21) are the only fixed points of the discrete scheme (4.11) or (4.18). The fact that the fixed point preserves the instability of equilibrium point $\bar{I} = 0$ and the local asymptotic stability of the equilibrium point $\bar{I} = K(1 - \frac{1}{\mathcal{R}_0})$ is obtained by Hartman-Grobman linearization theorem [23]. □

By using $K = 1000$, $I_0 = 10$, $\beta = 1.3$ and $\mu = \gamma = 0.2$ so that $\mathcal{R}_0 = 3.25$, Theorem 4.1 is illustrated by Fig 7. Equally the combined Fig 8 and Fig 9 support the theory in Theorems 4.2 and 4.3. It is apparent that $0 \leq I^k \leq N^k \leq K$ and that N^k tends to K monotonically. The discrete solution for $I(t)$ given by (4.1) is presented in Table 3 for different values of h which compared with Table 2 shows lower order of convergence.

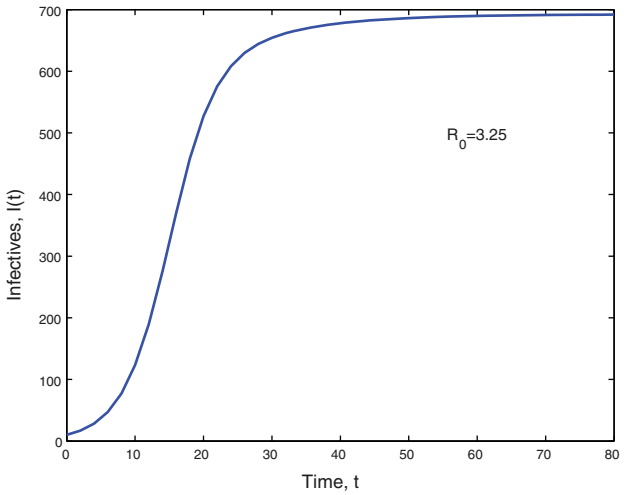


FIGURE 7. LAS endemic fixed-point for the NSFD scheme (4.4)-(4.5).

TABLE 3. NSFD scheme (4.1) with $c = 2$ and $\mathcal{R}_0 = 3.25$

Time t	*NSFD scheme*	*Exact solution*	*Error* : $h = 0.25$ $\phi = 0.2487$	*Error* : $h = 0.5$ $\phi = 0.4621$
$0.0000e+00$	$1.0000e+02$	$1.0000e+02$	$0.0000e+00$	$0.0000e+00$
$5.0000e+00$	$1.3753e+02$	$1.3787e+02$	$3.4169e-01$	$1.0362e+00$
$1.0000e+01$	$1.3845e+02$	$1.3845e+02$	$9.9151e-03$	$4.4197e-02$
$1.5000e+01$	$1.3846e+02$	$1.3846e+02$	$2.1713e-04$	$1.4954e-03$
$2.0000e+01$	$1.3846e+02$	$1.3846e+02$	$4.2943e-06$	$4.7623e-05$
$2.5000e+01$	$1.3846e+02$	$1.3846e+02$	$8.0825e-08$	$1.4867e-06$

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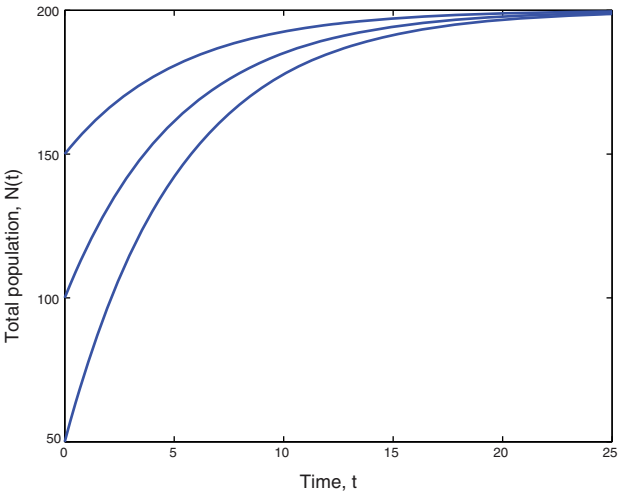


FIGURE 8. Positivity and boundedness of discrete solutions of the NSFD scheme (4.7).

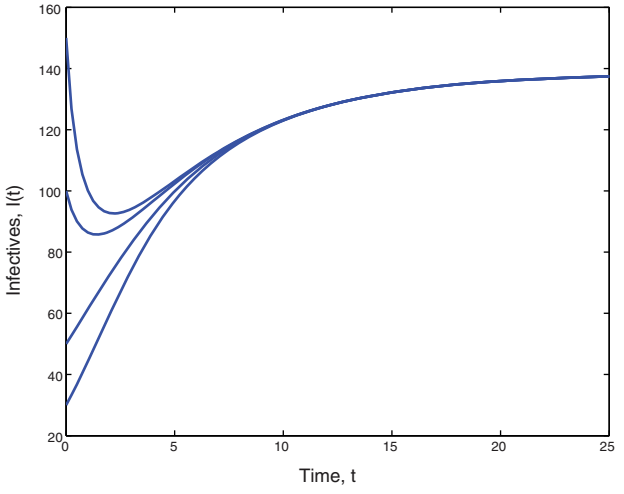


FIGURE 9. Elementary stability of the NSFD scheme (4.11).

5. NSFD schemes for SIS-Diffusion Model

Based on [**4**, **18**] and Section 4, we propose two NSFD schemes for the SIS-diffusion model (2.7)-(2.9).

The first NSFD scheme is

$$\frac{S_n^{k+1}-S_n^k}{\phi} = \mu K - \frac{\beta S^{k+1} I_n^k}{S_n^{k+1}+I_n^k} - \mu S_n^{k+1} + \gamma I_n^k + \frac{S_{n+1}^k - 2S_n^k + S_{n-1}^k}{\psi^2} \tag{5.1}$$

$$\frac{I_n^{k+1}-I_n^k}{\phi} = \frac{\beta S_n^{k+1} I_n^k}{S_n^{k+1}+I_n^k} - \mu I_n^{k+1} - \gamma I_n^k + \frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{\psi^2}, \tag{5.2}$$

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which leads to the NSFD scheme of the conservation law (2.9),

$$\frac{N_n^{k+1}-N_n^k}{\phi(\Delta t)} = \mu(K - N_n^{k+1}) + \frac{N_{n+1}^k - 2N_n^k + N_{n-1}^k}{\psi^2(\Delta x)}, \tag{5.3}$$

where

$$\psi(\Delta x) = \frac{2}{\sqrt{\beta(1-\frac{1}{\mathcal{R}_0})}} \sin\left(\frac{\sqrt{\beta(1-\frac{1}{\mathcal{R}_0})}}{2}\Delta x\right) \tag{5.4}$$

and $\phi(\Delta t)$ is given in (4.3). The denominator function in (5.4) is chosen in accordance with Mickens' rule for NSFD schemes regarding the stationary equation associated with (2.9) (see [**18**, **24**] for this specific case).

The second NSFD scheme for the SIS-diffusion model (2.9)-(2.10) is

$$\frac{N_n^{k+1}-N_n^k}{\phi(\Delta t)} = \mu\left(K - \frac{N_{n+1}^k + N_n^k + N_{n-1}^k}{3}\right) + \frac{N_{n+1}^k - 2N_n^k + N_{n-1}^k}{\psi^2(\Delta x)}, \tag{5.5}$$

$$\begin{aligned}\frac{I_n^{k+1}-I_n^k}{\phi(\Delta t)} &= \beta(1-\frac{1}{\mathcal{R}_0})\left(1 - \frac{I_n^{k+1}}{N_n^{k+1}(1-\frac{1}{\mathcal{R}_0})}\right)\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right)\\ &+ \frac{I_{n+1}^k - 2I_n^k + I_{n-1}^k}{(\psi(\Delta x))^2}.\end{aligned} \tag{5.6}$$

Subtracting (5.6) from (5.5) gives in view of (2.5) the corresponding NSFD scheme for (2.7):

$$\begin{aligned}\frac{S_n^{k+1}-S_n^k}{\phi(\Delta t)} &= \mu K - \frac{\beta S_n^{k+1}}{N_n^{k+1}}\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) - \mu\left(\frac{S_{n+1}^k + S_n^k + S_{n-1}^k}{3}\right)\\ &+ \gamma\left(\frac{I_{n+1}^k + I_n^k + I_{n-1}^k}{3}\right) + \left(\frac{S_{n+1}^k - 2S_n^k + S_{n-1}^k}{\psi^2(\Delta x)}\right),\end{aligned} \tag{5.7}$$

If we rearrange Equation (5.1), we get a quadratic equation in S_n^{k+1}:

$$A(S_n^{k+1})^2 + BS_n^{k+1} + C = 0, \tag{5.8}$$

where

$$\begin{aligned} A &= 1 + \mu\phi \geq 0,\\ B &= \left(1 + (\beta + \mu - \gamma)\phi\right) I_n^k - (S_n^k + \mu\phi K) - \frac{\phi}{\psi^2}\left(S_{n+1}^k - 2S_n^k + S_{n-1}^k\right)\\ \text{and}\quad C &= -\left[S_n^k + \gamma\phi I_n^k + \mu\phi K + \frac{\phi}{\psi^2}\left(S_{n+1}^k - 2S_n^k + S_{n-1}^k\right)\right] I_n^k. \end{aligned}$$

In what follows, we make the assumption

$$\frac{\phi}{\psi^2} = \frac{1}{2} \tag{5.9}$$

between the step sizes so that the term S_n^k cancels in the expressions of B and C.

Assuming that $S_n^k \geq 0$ and $I_n^k \geq 0$, then the unique positive solution of Equation (5.8) is given by

$$S_n^{k+1} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{5.10}$$

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Moreover, (5.2) and (5.3) rearranged into

$$I_n^{k+1} = \frac{\left(\frac{\beta\phi S_n^{k+1}}{S_n^{k+1}+I_n^k} + (1-\gamma\phi)\right) I_n^k + \frac{\phi}{\psi^2}\left(I_{n+1}^k - 2I_n^k + I_{n-1}^k\right)}{1+\mu\phi} \tag{5.11}$$

and

$$N_n^{k+1} = \frac{\mu\phi K + \frac{\phi}{\psi^2}(N_{n+1}^k + N_{n-1}^k) + (1-\frac{2\phi}{\psi^2})N_n^k}{1+\mu\phi} \tag{5.12}$$

show the positivity of I_n^{k+1} and N_n^{k+1} under the condition (4.17).

For the implementation of the second NSFD scheme, (5.5), (5.6) and (5.7) can be written as

$$N_n^{k+1} = \mu\phi K + \left(\frac{\phi}{\psi^2} - \frac{\mu\phi}{3}\right)\left(N_{n+1}^k + N_{n-1}^k\right) + \left(1 - \frac{\mu\phi}{3} - \frac{2\phi}{\psi^2}\right) N_n^k, \tag{5.13}$$

(5.14)

$$I_n^{k+1} = N_n^{k+1}(1-\frac{1}{\mathcal{R}_0})\frac{\frac{\phi}{\psi^2}(I_{n+1}^k + I_{n-1}^k) + \left(1-\frac{2\phi}{\psi^2}\right) I_n^k + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{N_n^{k+1}(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}$$

and

(5.15)

$$S_n^{k+1} = \frac{\mu\phi\left(K - \frac{S_{n+1}^k+S_n^k+S_{n-1}^k}{3}\right) + \gamma\phi\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right) + \frac{\phi}{\psi^2}\left(S_{n+1}^k - 2S_n^k + S_{n-1}^k\right) + S_n^k}{1 + \frac{\beta\phi}{N_n^{k+1}}\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}.$$

By using the equivalent formulations (5.12) and (5.13) of (5.3) and (5.5), we have the following result:

THEOREM 5.1. *Under the functional relation*

$$\frac{\phi}{\psi^2} = \frac{1}{3} \tag{5.16}$$

between the step sizes, the NSFD scheme (5.5) replicates the positivity and boundedness properties of the exact solution: $0 \le N_n^k \le K \Rightarrow 0 \le N_n^{k+1} \le K$. *The same conclusion holds for the NSFD scheme (5.3) under the condition (5.9).*

THEOREM 5.2. *Under the conditions of Theorem 5.1, the NSFD scheme (5.5)-(5.6) replicates specifically positivity and boundedness properties of the exact solution stated in Theorem 2.2 in the following manner:* $0 \le I_n^k \le N_n^k \le K \Rightarrow 0 \le I_n^{k+1} \le N_n^{k+1} \le K$.

PROOF. By using (5.16), Equations (5.14) and (5.15) are reduced into

$$I_n^{k+1} = N_n^{k+1}(1-\frac{1}{\mathcal{R}_0})\frac{\frac{1}{3}(I_{n+1}^k + I_n^k + I_{n-1}^k) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{N_n^{k+1}(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} \tag{5.17}$$

and

(5.18)

$$S_n^{k+1} = \frac{\mu\phi\left(K - \frac{S_{n+1}^k+S_n^k+S_{n-1}^k}{3}\right) + \gamma\phi\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right) + \frac{1}{3}\left(S_{n+1}^k + S_n^k + S_{n-1}^k\right)}{1 + \frac{\beta\phi}{N_n^{k+1}}\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}.$$

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For $0 \leq I_n^k,\ S_n^k \leq N_n^k \leq K$, Theorem 5.1 and Equations (5.17)-(5.18), imply that $N_n^{k+1} \geq 0$; $I_n^{k+1} \geq 0$ and $S_n^{k+1} \geq 0$. Thus, from the time-space analogue of (4.10), we have $0 \leq I_n^{k+1},\ S_n^{k+1} \leq N_n^{k+1} \leq K$. □

Theorem 5.2 is of course valid for traveling wave solution when $N = K$ is constant. More importantly, in this case, we have the next result where the condition (5.16) between step sizes is more relaxed.

THEOREM 5.3. *Under the functional relation (5.9), the NSFD scheme (5.5)-(5.6) replicates the positivity and boundedness properties of the exact solution as follows:*

$$0 \leq I_n^k \leq K(1-\frac{1}{\mathcal{R}_0}) \Rightarrow 0 \leq I_n^{k+1} \leq K(1-\frac{1}{\mathcal{R}_0}), \tag{5.19}$$

and

$$K(1-\frac{1}{\mathcal{R}_0}) \leq I_n^k \leq K \Rightarrow\ K(1-\frac{1}{\mathcal{R}_0}) \leq I_n^{k+1} \leq K. \tag{5.20}$$

PROOF. In view of (5.9), Equation (5.14) becomes

$$I_n^{k+1} = K(1-\frac{1}{\mathcal{R}_0})\frac{\frac{1}{2}(I_{n+1}^k + I_{n-1}^k) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}. \tag{5.21}$$

If $0 \leq I_n^k \leq K(1-\frac{1}{\mathcal{R}_0})$ so that

$$\frac{I_{n+1}^k + I_{n-1}^k}{2} \leq K(1-\frac{1}{\mathcal{R}_0}), \tag{5.22}$$

we have from (5.22)

$$\frac{\frac{1}{2}\left(I_{n+1}^k + I_{n-1}^k\right) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} \leq 1,$$

which in view of (5.21) gives (5.19).

Assume now that $K(1-\frac{1}{\mathcal{R}_0}) \leq I_n^k \leq K$ so that $K(1-\frac{1}{\mathcal{R}_0}) \leq \frac{1}{2}(I_{n+1}^k + I_{n-1}^k) \leq K$. Then

$$\begin{aligned} K(1-\frac{1}{\mathcal{R}_0}) &= K(1-\frac{1}{\mathcal{R}_0})\frac{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} \\ &\leq K(1-\frac{1}{\mathcal{R}_0})\frac{\frac{1}{2}(I_{n+1}^k + I_{n-1}^k) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)}{K(1-\frac{1}{\mathcal{R}_0}) + \phi\beta(1-\frac{1}{\mathcal{R}_0})\left(\frac{I_{n+1}^k+I_n^k+I_{n-1}^k}{3}\right)} = I_n^{k+1} \leq K \end{aligned}$$

by (5.21). □

To conclude this section, we illustrate the dynamical consistency of the NSFD schemes (5.3) and (5.6) with respect to positivity and boundedness, as stated in Theorem 5.1 (Fig 10) and Theorem 5.3 (Fig 11), respectively. In both examples, we took $\beta = 1.3,\ \mu = \gamma = 0.2,\ K = 100$ so that $\mathcal{R}_0 = 3.1$. The initial conditions are taken to be $N_0(x) = 20 + 10\sin(2\pi x/5)$ and $I_0(x) = 10 + 10\sin(2\pi x/5)$.

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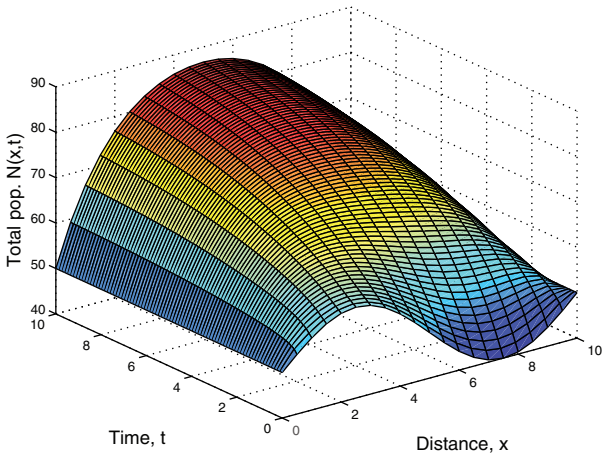


FIGURE 10. Positivity and boundedness solutions for the NSFD scheme (5.3).

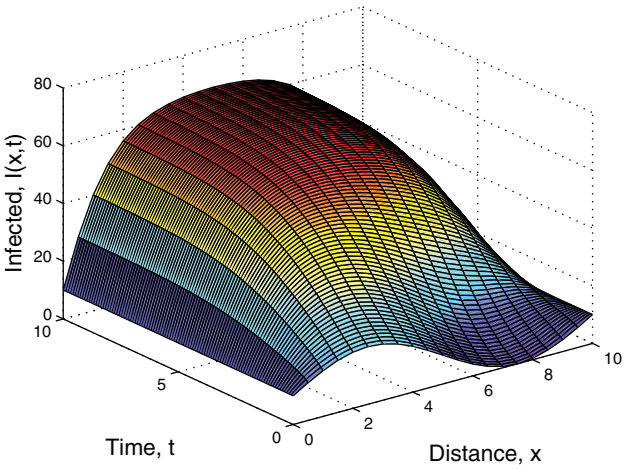


FIGURE 11. Positivity and boundedness solutions for the NSFD scheme (5.6).

6. Conclusion

Although the nonstandard finite difference (NSFD) method was introduced more than two decades ago, the important issue of investigating higher order such methods has not been addressed. In this work, we have filled this gap by designing nonstandard Runge-Kutta methods of order $k \geq 1$. This is done in the framework of the SIS epidemiological model. In particular, we extended the classical (explicit) 4 stage Runge-Kutta method to a nonstandard scheme, which is shown to be theoretically and computationally of order four.

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In view of the Holling type II or Michaelis-Menten function response that occurs in the SIS model with or without diffusion, the study is taken one step further to design innovative NSFD schemes, which add great value to existing schemes [**3**, **9**, **16**, **19**] by faithfully replicating the dynamics of the continuous model such as the global asymptotic stability of the disease-free equilibrium, the positivity and the boundedness of the solutions.

Our ongoing research includes the extension of this work to more general reaction-diffusion SIS epidemic models in which the contact and recovery rates depend on the space variables, while the diffusion coefficients are different (see [**1**, **2**, **20**]). We are also busy designing NSFD schemes that capture the latent period in epidemiological delay differential equations [**11**] and the period of infectivity in epidemiological Volterra integral equations [**8**].

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