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APPROXIMATING FIXED POINTS OF GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract

Under certain conditions the convergence of Ishikawa iterate to a unique fixed point is proved for nonexpansive type mappings in a uniformly convex Banach space. In this paper we improve the results given by Pathak and Khan [6].

1. Introduction

Let D be a nonempty, closed and convex subset of a uniformly convex Banach space B, and $T : D \to D$ with fixed point set F(T). Recently Ghosh and Debnath [1] introduced the generalized versions of the condtions of Senter and Dotson [7] as the mapping T with $F(t) \neq \phi$ is said to satisfy the following conditions:

Condition A : If there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(\gamma) > 0$ for all $\gamma \in (0, \infty)$ such that

$$\| (1 - TT_\mu)x \| \geq f(d(x, F)) \quad \forall x \in D, \tag{1.1}$$

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where $T_\mu x = (1-\mu)x + \mu Tx$ with $0 \leq \mu \leq \beta < 1$ and

$$d(x, F) = inf_{z \in F} \parallel x - z \parallel .$$

Condition B : If there exists a positive real number k such that

$$\parallel (1 - TT_\mu)x \parallel \geq kd(x, F(T)) \quad \forall x \in D. \tag{1.2}$$

When $\mu = 0$, both conditions reduce to those of Senter and Dotson [7]. It may be noted that the mapping satisfying Condition (B) also satisfies Condition (A). In this paper these conditions are used to prove the convergence of Ishikawa iterates [3] of certain generalized nonexpansive mappings.

2. Ishikawa's Iterative Process

Let D be a convex subset of Banach space B and $T : D \to D$. For $x_1 \in D$, Ishikawa [3] defined a sequence $\{x_n\}$ such that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n Tx_n], \tag{2.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequence of nonnegative numbers with $0 \leq \alpha_n \leq \beta_n \leq 1$,

$$\lim_{n \to \infty} \beta_n = 0$$

and

$$\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty.$$

Using notation for $T_\mu x$ of Section 1, (2.1) may be written as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n TT_{\beta_n} x_n. \tag{2.2}$$

In this paper, we assume that α_n and β_n satisfy
(i) $0 < a \leq \alpha_n < b < 1$,
(ii) $0 \leq \beta_n \leq \beta < 1$.
We denote the sequence (2.1) by $M(x_1, \alpha_n, \beta_n, T)$, where α_n and β_n satisfy (i) and (ii). We also assume that $\alpha_n = \lambda$ and $\beta_n = \mu$ for all n in the Ishikawa iterates defined above, that is,

$$x_{n+1} = T^n_{\lambda,\mu} x_1, \quad T_{\lambda,\mu} = (1 - \lambda)I + \lambda T[(1 - \mu)I + \mu T]. \tag{2.3}$$

3. Generalized Nonexpensive Mappings and Convergence Theorems

Let us recall some definitions which are useful for our results.

Definition 3.1 : A mapping $T : D \to D$ is called nonexpansive if for all $x, y \in D$,

$$\| Tx - Ty \| \leq \| x - y \| . \tag{3.1}$$

Whereas $T : D \to D$ is said to be quasi-nonexpansive if for all $x \in D$,
$F(T) \neq \phi$ and $p \in F(T)$ where

$$\| Tx - p \| \leq \| x - p \| . \tag{3.1}'$$

Definition 3.2 : A mapping $T : D \to D$ is called generalized nonexpansive if it satisfies the condition, for all $x, y \in D$,

$$\| Tx - Ty \| \leq a \| x - y \| + b \{ \| x - Tx \| + \| y - Ty \| \}$$

$$+ c \{ \| x - Ty \| + \| y - Tx \| \} , \tag{3.2}$$

where $a, c \geq 0, b > 0$ and $a + 2b + 2c \leq 1$. This type of mapping was introduced by Hardy and Rogers [2] in metric spaces.

Definition 3.3 : A mapping $T : D \to D$ is said to satisfy Condition (A) if for all $x, y \in D$,

$$\| Tx - Ty \| \leq max \left\{ \beta \| x - y \|, \frac{\| x - Tx \| + \| y - Ty \|}{2}, \frac{\| x - Ty \| + \| y - Tx \|}{2} \right\}, \tag{3.3}$$

whereas T is said to satisfy Condition (A) if for all $x, y \in D$,

$$\| Tx - Ty \| \leq max \left\{ \beta \| x - y \|, \sqrt{\| x - Tx \| . \| y - Ty \|}, \sqrt{\| x - Ty \| . \| y - Tx \|} \right\}, \tag{3.3}'$$

and T is said to satisfy Condition (B) if for all $x, y \in D$,

$$\| Tx - Ty \| \leq max \left\{ \beta \| x - y \|, \frac{\| x - Tx \| + \| y - Ty \|}{2}, \| x - Ty \|, \beta \| y - Tx \| \right\}, \tag{3.4}$$

whereas T is said to satisfy Condition (B) if for all $x, y \in D$,

$$\| Tx - Ty \| \leq max \left\{ \beta \| x - y \|, \sqrt{\| x - Tx \| . \| y - Ty \|}, \| x - Ty \|, \beta \| y - Tx \| \right\}, \tag{3.4}'$$

where $0 \leq \mu \leq \beta < 1$.

Note that $(3.3)' \Rightarrow (3.3)$ and $(3.4)' \Rightarrow (3.4)$.

Remark 3.4 It is to be noted that

(i) a nonexpansive mapping is generalized nonexpansive,

(ii) generalized nonexpansive mappings and mappings satisfying Condition (A) also satisfy condition (B), but converse is not true as shown under:

$$\begin{array}{ccc} (3.3) & \Longrightarrow & (3.4) \\ & \not\Longleftarrow & \\ \Downarrow\Uparrow & & \Downarrow\Uparrow \\ (3.3)' & \Longrightarrow & (3.4)' \\ & \not\Longleftarrow & \end{array}$$

For this we give the following example:

Example 3.5 Let B = R with the usual norm and let $D = D_1 \bigcup D_2$ where

$D_1 = \frac{m}{n}, m = 0, 1, 3, ..., n = 1, 4, ...3k + 1,$

$D_2 = \frac{m}{n}, m = 1, 3, 9, 27, ..., n = 2, 5, ...3k + 2.$ (3.5)

Define $T : D \to D$ by

$$Tx = \begin{cases} \frac{3x}{4}, & x \in D_1, \\ \\ \frac{x}{2}, & x \in D_2. \end{cases} \tag{3.6}$$

Then T satisfy Condition (B), but it does not satisfy Condition (A) and coincidentally that T is not a generalized nonexpansive mapping. For instance, take $x = 1, y = 3/5$. Then

$$\| Tx - Ty \| = \frac{9}{20} \geq max \left\{ \frac{2}{5}\beta, \frac{11}{40}, \frac{17}{40} \right\}$$

$$= max \left\{ \frac{2}{5}\beta, \frac{1}{2}[\frac{1}{4} + \frac{3}{10}], \frac{1}{2}[\frac{7}{10} + \frac{3}{20}] \right\}$$

$$= max \left\{ \beta \| x - y \|, \frac{\| x - Tx \| + \| y - Ty \|}{2}, \frac{\| x - Ty \| + \| y - Tx \|}{2} \right\} \tag{3.7}$$

We now show that a mapping T with $F(T) \neq \phi$ satisfying Condition (B) is a quasi-nonexpansive mapping. Suppose p is a fixed point of T. Then putting y = p in (3.4)'

and for $Tx \neq p$, we obtain

$$\begin{aligned} 0 <\| Tx - p \| &= \| Tx - Tp \| \\ &\leq max\{\beta \| Tx - p \|, \sqrt{\| x - Tx \| . \| p - p \|}, \| x - p \|, \beta \| p - Tx \|\} \\ &\leq max\{\beta \| Tx - p \|, 0, \| x - p \|, \beta \| p - Tx \|\}. \end{aligned} \tag{3.8}$$

Since $\| Tx - p \| \leq \beta \| p - Tx \|$ is not possible, we have

$$\| Tx - p \| \leq max\{\beta \| x - p \|, \| x - p \|\} \tag{3.9}$$

which implies that

$$\| Tx - p \| \leq \| x - p \| . \tag{3.10}$$

Therefore, T is quasi-nonexpansive. Next we show that

$$F(T) = F(T_{\lambda,\mu}) = F(TT_{\mu}). \tag{3.11}$$

Obviously $F(T) \subset F(T_{\lambda,\mu})$. Let $p \in F(T_{\lambda,\mu})$. Then $T_{\lambda,\mu}p = p$ implies that

$$T_{\lambda,\mu}p = (1-\lambda)I_p + \lambda T[(1-\mu)I_p + \mu T_p].$$

$= (1-\lambda)p + \lambda TT_{\mu}p$ and so $TT_{\mu}p = p$.

It follows from (3.4)' that

$$\begin{aligned} \| Tp - p \| &= \| Tp - T(T_{\mu}p) \| \\ &\leq max\{\beta \| p - T_{\mu}p \|, \sqrt{\| p - Tp \| . \| T_{\mu}p - p \|}, 0, \beta \| T_{\mu}p - Tp \|\} \\ &\leq max\{\beta \| p - T_{\mu}p \|, \sqrt{\mu} \| p - Tp \|, 0, \beta \| p - Tp \|\} \end{aligned} \tag{3.12}$$

and $Tp = p$, since $max\{\beta\mu, \frac{1}{2}(1+\mu), \beta(1-\mu)\} < 1$.

Thus, $F(T_{\lambda,\mu}) \subset F(T)$ leading to the result (3.11).

Now, we show that the mapping T satisfies Condition (B). We have from (3.4)'

$$\begin{aligned} \| TT_{\mu}x - p \| &= \| TT_{\mu}x - Tp \| \\ &\leq max\{\beta \| T_{\mu}x - p \|, \sqrt{\| p - Tp \| . \| T_{\mu}x - TT_{\mu}x \|}, \| T_{\mu}x - p \|, \beta \| p - TT_{\mu}x \|\} \\ &\leq max\{\| T_{\mu}x - p \|, \beta \| p - TT_{\mu}x \| \end{aligned}$$

$$
\begin{aligned}
&= \| T_\mu x - p \| \\
&\leq [(1-\mu) \| x - p \| + \mu \| Tx - p \|] \\
&\leq [(1-\mu) \| x - p \| + \mu \| x - p \| \\
&\leq \| x - p \| . \qquad (3.13)
\end{aligned}
$$

Also we know that

$$\| TT_\mu x - p \| \geq \| x - p \| - \| x - TT_\mu x \| \qquad (3.14)$$

which implies that

$$\| x - TT_\mu x \| \geq 0 = f(0).$$

Thus T satisfies Condition (A). Consequently, by Maiti and Ghosh [1] we have the following:

Theorem 3.6 : Let D be a closed convex subset of a Banach space B and let $T : D \to D$ be a mapping which satisfies (3.4) and has a fixed point in D. Then T satisfies Conditon (B) and for any $x_1 \in D, M(x_1, \alpha_n, \beta_n, T)$ converges to the fixed point of D.

A variant of Theorem 3.6 is stated below for a mapping T satisfying Condition (A).

Theorem 3.7 : Let D be a closed convex bounded subset of a uniformly convex Banach space B, and let $T : D \to D$ be a mapping satisfying (3.3). Then T satisfies Condition (A) and for any $x_1 \in D, M(x_1, \alpha_n, \beta_n, T)$ converges to the unique fixed point of T.

Proof : The mapping T satisfying (3.3) also satisfies Naimpally and Singh [5, Condition II(D)], and so T has a unique fixed point. We now show that T is quasi nonexpansive. Let $p \in F(T)$. Then for any $x \in D$, we have from (3.3)',

$$\| Tx - p \| = \| Tx - Tp \| \leq max\{\beta \| x - p \|, 0, \sqrt{\| x - p \| . \| p - Tx \|} \qquad (3.15)$$

implying thereby

$$\| Tx - p \| \leq \| x - p \| . \qquad (3.16)$$

Next, we show that T satisfies Condition (A).

Let $p \in F(T)$. Then we have from (3.1),

$$
\begin{aligned}
&\| TT_\mu x - p \| = \| TT_\mu x - Tp \| \\
&\leq max\{\beta \| T_\mu x - p \|, 0, \sqrt{\| T_\mu x - p \| . \| p - TT_\mu x \| \,|}\}
\end{aligned}
$$

$$\leq max\{\beta \parallel T_\mu x - p \parallel, \parallel T_\mu x - p \parallel\}$$

$$= \parallel T_\mu x - p \parallel = \parallel x - p \parallel . \tag{3.17}$$

From (3.14) and ((3.17), we derive

$$\parallel x - p \parallel \geq \parallel x - p \parallel - \parallel x - TT_\mu x \parallel \tag{3.18}$$

which implies that

$$\parallel x - TT_\mu x \parallel \geq 0 = f(0). \tag{3.19}$$

Thus T satisfies all conditions which ensure the convergence of $M(x_1, \alpha_n, \beta_n, T)$.

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