$(\alpha, \beta) - L^p$ 2-Norm Orthogonality and Characterizations of 2 - Inner Product Spaces

Vinai K. Singh, S. Kumar* and A. K. Singh**

Department of Mathematics,
R.D. Engineering College,
N.H. 58 Delhi Meerut Road,
Duhaí, Ghaziabad, INDIA.
*Department of Applied Mathematics
Inderprastha Engineering College,
Ghaziabad 201010, INDIA.
vinaiksingh@rediffmail.com; drsengar2002@yahoo.co.uk

**Department of Science and Technology,
Technology Bhawan, New Mahrauli Road,
New Delhi-100016, INDIA.

ABSTRACT

In the present paper we have characterised $(\alpha, \beta) - L^p$ orthogonality in a 2-normed linear space. In some way the results proved in this paper generalize some of the similar characterization of generalized $L^p$- orthogonality derived earlier by Zheng Liu[8].

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INTRODUCTION

Recently there has been special interest to deal with certain analytic functional aspects in 2-normed spaces of finite or infinite dimensional type.
Usually orthogonality is dealt in inner product spaces but there is a concept like orthogonality in normal linear spaces ([2], [3], [5], [6], [7] and [8]). As has been noted earlier (for example see reference [7]) Birkhoff orthogonality plays a typical role in a normed linear space. In some analytic consideration also Birkhoff orthogonality is important.

In the present paper we have introduced \((x, \alpha, \beta) - L^p \perp\) -orthogonality for a pair \((x, z)\) and \((y, z)\) in 2-normed spaces. We have also developed certain properties in the line of those given earlier by Liu [8] as was given in the normed spaces to be carried over in the setting of 2-normed space and 2-inner product spaces.

**PRELIMINARIES AND NOTATIONS**

**DEFINITION 1.** Let \(p > 1\) be a fixed real number. If \((x, z) \in X \times X\), then we say that \((x, z)\) is \(L^p\)-orthogonal and we denote \((x, z) \perp_{L^p} (y, z)\) provided \[\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p\] is called left \(L^p\) orthogonality. In a similar way \((x, y) \perp_{L^p} (x, z)\) provided \[\| x, y + z \|^p = \| x, y \|^p + \| x, z \|^p\] is called right \(L^p\)-orthogonality in 2-normed spaces.

**DEFINITION 2.** Let \(p \geq 1\) and \(\alpha, \beta \neq 1\) be fixed real numbers. If \((x, z) \in X \times X\) then \((x, z)\) is 2-norm \((\alpha, \beta) - L^p\) -orthogonal to \((y, z)\) denote by \((x, z) \perp_{L^p} (y, z)(\alpha, \beta)\) provided that \[\| x + y, z \|^p + \| \alpha x + \beta y, z \|^p = \| x, z \|^p + \| \alpha x + \beta y, z \|^p\]

and \(z \notin V(x, y)\) (where \(V(x, y)\) is the linear span of \(x, y \in X\)). Similarly we say \((x, z)\) is \((\alpha, \beta) - L^p\) -orthogonal to \((y, z)\) denoted by \((x, z) \perp_{L^p} (y, z)(\alpha, \beta)\) provided that \[\| x, y + z \|^p + \| x, \alpha y + \beta z \|^p = \| x, \alpha y + z \|^p + \| x, y + \beta z \|^p\].

**LEMMA 1.** For all \((x, z), (y, z) \in X \times X\), \(\alpha, \beta \neq 1\), \((x, z) \perp_{L^p} (y, z)(\alpha, \beta)\) if and only \((y, z) \perp_{L^p} (x, z)(\alpha, \beta)\). The following theorem and corollary demonstrate that the concept of \((\alpha, \beta) - L^p\) - orthogonality is non-vacuous.
THEOREM 1. Let $p > 1$ and $\alpha, \beta \neq 1$ be a fixed real numbers. If $(x, z) \neq 0, (y, z) \in X \times X$ then there exists a real number such that $((x, z), \perp_L (ax + y, z))(\alpha, \beta)$.

PROOF. Set

$$f(t) = \| x + \alpha x + y, z \|_p^p + \| \alpha x + \beta (tx + y), z \|_p^p - \| \alpha x + tx + y, z \|_p^p - \| x + \beta (tx + y), z \|_p^p.$$ 

Clearly, $f$ is a continuous function on $-\infty < t < \infty$, and we have, for $t \neq 0$

$$f(t) = |t|^p [\| x + \frac{\alpha x + y}{t}, z \|_p^p - \| x, z \|_p^p] - (\| x + \frac{\alpha x + y}{t}, z \|_p^p - \| x, z \|_p^p) - (\| \beta x + \frac{x + \beta y}{t}, z \|_p^p - \| \beta x, z \|_p^p).$$

Then for $t \neq 0$

$$\frac{f(t)}{|t|^{p-1} \sgn t} = \| x + \frac{1}{t} (x + y), z \|_p^p - \| x, z \|_p^p + \| \beta x + \frac{1}{t} (\alpha x + \beta y), z \|_p^p - \| \beta x, z \|_p^p,$$

and hence

$$\lim_{t \to +\infty} \frac{f(t)}{|t|^{p-1} \sgn t} = p \| x, z \|_p^{p-1} J_+ (x, z)(x+y) + p \| \beta x, z \|_p^{p-1} J_+ (\beta x, z)(\alpha x + \beta y) - p \| x, z \|_p^{p-1} J_+ (x, z)(x+y) - p \| \beta x, z \|_p^{p-1} J_+ (\beta x, z)(x+y),$$

where $J_+ ((x, z)(y, z))$ and $J_-((x, z)(y, z))$ are respectively the right and left Gateaux derivative of the norm at $(x, z)$, keeping second co-ordinate as fixed in the direction of $(y, z)$. By James [5] we see that

$$J_+ (x, z)(r x + s y) = r \| x, z \|_p + s J_+ (x, z)(y, z)$$

for some $s \geq 0$ and $r$, therefore,

$$\lim_{t \to +\infty} \frac{f(t)}{p - 1} = p \| x, z \|_p^p (1 + \alpha \beta^{p-1} - \alpha - \beta^{p-1})$$

$$p \| x, z \|_p^p (1 - \alpha)(1 - \beta^{p-1}).$$
Thus for any fixed real number $\alpha, \beta \neq 1$ we have either $f(t) \to \infty$ as $t \to +\infty$ or $f(t) \to -\infty$. Hence there is a real number a such that $f(a) = 0$, which was to be proved.

**COROLLARY 1.** Let $p > 1$ and $\alpha, \beta \neq 1$ be the fixed real numbers. If $x \neq 0, z \neq 0, (y, z) \in X$, then there exist a real number a such that 
\[ ((ax + yz) \perp_L (x, z)(\alpha, \beta)) \]

**PROOF.** The result follows from Theorem 1 and Lemma 1.

**Lemma 2.** Let $(x, z)$ or $(y, z) \in X \times X$ and $\alpha, \beta \neq 1$

(i) $\alpha, \beta \neq 0, ((x, z) \perp_L (y, z))(\alpha, \beta)$ if and only if $((\alpha x, z) \perp_L (\beta y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,

(ii) if $\beta \neq 0, ((x, z) \perp_L (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_L (\beta y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,

(iii) if $\alpha \neq 0, ((x, z) \perp_L (y, z))(\alpha, \beta)$ if and only if $((\alpha x, z) \perp_L (\beta y, z))(\frac{1}{\alpha}, \beta)$.

Homogeneity, symmetry and left and right additivity of $(\alpha, \beta) - L^p$-orthogonality are defined in usual way, i.e. 2-norm $(\alpha, \beta) - L^p$-orthogonality is homogeneous provided for all $x, y, z \in X$ and real numbers $a, b, ((x, z) \perp_L (y, z))(\alpha, \beta)$ implies $((ax, z) \perp_L (by, z))(\alpha, \beta)$; 2-norm $(\alpha, \beta) - L^p$-orthogonality is symmetric provided for all $x, y, z \in X, ((x, z) \perp_L (y, z))(\alpha, \beta)$ implies $(y, z) \perp_L (x, z))(\alpha, \beta)$; 2-norm $(\alpha, \beta) - L^p$-orthogonality is left additive if and only if for all $x, y, w \in X, ((x, z) \perp_L (w, z))(\alpha, \beta)$ and $(y, z) \perp_L (w, z))(\alpha, \beta)$ implies $((x + y, z) \perp_L (w, z))(\alpha, \beta)$ and 2-norm $(\alpha, \beta) - L^p$-orthogonality is right additive if and only if for all $x, z, (y, z), (w, z) \in X \times X, ((x, z) \perp_L (y, z))(\alpha, \beta)$ and if $(y, z) \perp_L (w, z))(\alpha, \beta)$ imply $((x, z) \perp_L (y + w, z))(\alpha, \beta)$.

The following two corollaries are immediate consequences of the definition of homogeneity and Lemma 1 and Lemma 2.

**COROLLARY 2.** For all $\alpha, \beta \neq 1, 2$-norm $(\alpha, \beta) - L^p$-orthogonality is homogeneous if and only if 2-norm $(\alpha, \beta) - L^p$-orthogonality is homogeneous.

**COROLLARY 3.** Suppose 2-norm $(\alpha, \beta) - L^p$-orthogonality is homogeneous.

(i) If $\alpha, \beta \neq 0, ((x, z) \perp_L (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_L (y, z))(\frac{1}{\alpha}, \frac{1}{\beta})$,

(ii) if $\beta \neq 0, ((x, z) \perp_L (y, z))(\alpha, \beta)$ if and only if $((x, z) \perp_L (y, z))(\alpha, \frac{1}{\beta})$. 
(iii) if $\alpha \neq 0$, $(x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ if and only if $(x, z) \perp_{L^p} (y, z))(1, \beta)$.

Now let us further study some consequences of homogeneity.

**Lemma 3.** If $\alpha \neq -1$ and $(\alpha, \beta) - L^p - 2$-norm orthogonality is homogeneous, then $(x, z) \perp_{L^p} (y, z))(\alpha, \beta)$ implies

$$
\|x + y, z\|^p = (1 - |\beta|^p) \|y, z\|^p + \|x + \beta y, z\|^p.
$$

**Proof.** From Corollary 3, it suffices to assume $|\alpha| < 1$. Suppose $(x, z) \perp_{L^p} (y, z))(\alpha, \beta)$. Then $\|x + y, z\|^p + \|\alpha x + \beta y, z\|^p = \|x + \beta y, z\|^p$ keeping second co-ordinate as fixed. Since the result is immediate for $\alpha = 0$. We may assume that $\alpha \neq 0$. We are denoting the statement by $P(n)$ i.e.

$$P(n) : \|x + y, z\|^p + \|\alpha^n x + \beta y, z\|^p = \|\alpha^n x + y, z\|^p + \|x + \beta y, z\|^p.$$

Clearly $P(1)$ is true and if $P(n)$ is true for some positive integer $n$. Since 2-norm $(\alpha, \beta) - L^p -$ orthogonality is homogeneous, $(\alpha^n x, z) \perp_{L^p} (y, z))(\alpha, \beta)$, we have

$$\|\alpha^n x + y, z\|^p + \|\alpha^{n+1} x + \beta y, z\|^p = \|\alpha^{n+1} x + y, z\|^p + \|\alpha^n x + \beta y, z\|^p.$$

Adding this to $P(n)$ we obtain

$$\|x + y, z\|^p + \|\alpha^{n+1} x + \beta y, z\|^p = \|\alpha^{n+1} x + y, z\|^p + \|x + \beta y, z\|^p,$$

which is $P(n + 1)$. Thus $P(n)$ is true for all positive integer $n$, but

$$\lim_{n \to \infty} \alpha^n = 0,$$

so in the limit, by continuity of the norm, we have

$$\|x + y, z\|^p + |\beta|^p \|y, z\|^p = \|y, z\|^p + \|x + \beta y, z\|^p,$$

and the conclusion of the lemma follows.

**Theorem 2.** If $\alpha, \beta \neq -1$ and 2-norm $(\alpha, \beta) - L^p -$ orthogonality is homogeneous, then $(x, y) \perp_{L^p} (y, z))(\alpha, \beta)$ implies

$$\|x + y, z\|^p = \|x, z\|^p + \|y, z\|^p.$$
i.e. 2 - norm \((\alpha, \beta) - L^p\) orthogonality implies \(L_p\) orthogonality.

**PROOF.** By Corollary 6, we may assume \(|\beta| < 1\). Suppose \(((x, z) \perp_{L^p} (y, z))(\alpha, \beta)\) and let \(Q(n)\) denote the statement

\[
Q(n) : \| x + y, z \|^p = (1 - |\beta^n|^p) \| y, z \|^p + \| x + \beta^n y, z \|^p.
\]

The statement \(Q(1)\) is Lemma 3. If we assume \(Q(n)\) is true for some positive integer \(n\), since \(((x, z) \perp_{L^p} (y, z))(\alpha, \beta)\) by homogeneity, we have by Lemma 3,

\[
\| x + \beta^n y, z \|^p = (1 - |\beta|^p) \| \beta^n y, z \|^p + \| x + \beta^{n+1} y, z \|^p
\]

Substituting this in \(Q(n)\) we obtain

\[
\| x + y, z \|^p = (1 - |\beta^{n+1}|) \| y, z \|^p + \| x + \beta^{n+1} y, z \|^p
\]

or \(Q(n+1)\).

Hence \(Q(n)\) holds for all positive integer \(n\). Since \(\beta^n \to 0\) as \(n \to +\infty\) by taking limit in \(Q(n)\) we obtain

\[
\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p.
\]

**THEOREM 3.** If 2 - norm \((\alpha, \beta) - L^p\) orthogonality is homogeneous, then \(((x, z) \perp_{L^p} (y, z))(\alpha, \beta)\) implies \(\| x - y, z \| = \| x + y, z \|\) i.e. 2 - norm \((\alpha, \beta) - L^p\) orthogonality implies 2 - norm isosceles orthogonality.

**PROOF.** If \((\alpha, \beta) = -1\) the result is follows. Otherwise by Lemma 1 we may assume without loss of generality that \(\alpha \neq -1\). If \(\beta = -1\) the result is immediate from Lemma 7. If \(\beta \neq -1\), by Theorem 4, we have

\[
\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p.
\]

But by homogeneity \(((x, z) \perp_{L^p} (y, z))(\alpha, \beta)\) holds, so \(\| x - y, z \|^p = \| x, z \|^p + \| -y, z \|^p\) and the result follows.

**LEMMA 4.** For all \(\alpha, \beta \neq 1\), each one of the following:

(i) \((\alpha, \beta) - L^p\) 2 - norm orthogonality is symmetric and left additive,

(ii) \((\alpha, \beta) - L^p\) 2 - norm orthogonality is symmetric and right additive,

(iii) \((\alpha, \beta) - L^p\) 2 - norm orthogonality is left and right additive,

implies that \((\alpha, \beta) - L^p\) 2 - norm orthogonality is homogeneous.
PROOF. Suppose (i) holds and \(((x, z) \perp_{L^p} (y, z)) (\alpha, \beta)\), where \(x\), \(y\) and \(z\) are arbitrary elements. Since the result is obvious for \(x = 0\) or \(y = 0\). We will assume \(x \neq 0\) and \(y \neq 0\). By Corollary 1, there exist a real number \(a\) such that \(((ay - x, z) \perp_{L^p} (y, z)) (\alpha, \beta)\). Left additive then gives \(((ay, z) \perp_{L^p} (y, z)) (\alpha, \beta)\) and hence \(a = 0\). Thus \(((x, z) \perp_{L^p} (y, z)) (\alpha, \beta)\), using left additive and symmetry, we find it now follows that \(((nx, z) \perp_{L^p} (my, z)) (\alpha, \beta)\) for all integers \(m\) and \(n\), i.e.

\[
\| nx + my, z \|_{L^p} = \| \alpha nx + \beta my, z \|_{L^p} = \| \alpha x + \beta m y, z \|_{L^p} + \| nx + \beta my, z \|_{L^p}
\]

or

\[
\| x + \frac{m}{n} y, z \|_{L^p} = \| \alpha x + \beta \frac{m}{n} y, z \|_{L^p} = \| \alpha x + \beta \frac{m}{n} y, z \|_{L^p} + \| x + \beta \frac{m}{n} y, z \|_{L^p}.
\]

From the continuity of the norm it follows that

\[
\| x + ky, z \|_{L^p} = \| \alpha x + \beta ky, z \|_{L^p} = \| \alpha x + ky, z \|_{L^p} + \| x + ky, z \|_{L^p}
\]

for all real numbers \(k\) or \(((x, z) \perp_{L^p} (ky, z)) (\alpha, \beta)\) for all \(k\). So \((\alpha, \beta) - L^p - 2\) - norm orthogonality is homogeneous. By similar reasoning we can also get that (ii) and (iii) imply \((\alpha, \beta) - L^p - 2\) - norm orthogonality is homogeneous.

By similar reasoning, we can also get that (ii) and (iii) imply \((\alpha, \beta) - L^p - 2\) - norm orthogonality is homogeneous.

The result may be summarized as follows:

THEOREM 4. Let \(p > 1\) and \(\alpha, \beta \neq 1\). The following are equivalent.

(i) \((\alpha, \beta) - L^p - 2\) - norm orthogonality is homogeneous,

(ii) \((\alpha, \beta) - L^p - 2\) - norm orthogonality is symmetric and left additive,

(iii) \((\alpha, \beta) - L^p - 2\) - norm orthogonality is symmetric and right additive,

(iv) \((\alpha, \beta) - L^p - 2\) - norm orthogonality is left and right additive.

Finally we give two characterizations of inner product spaces based on the relation between \((\alpha, \beta) - L^p, 2\) - norm orthogonality and Birkhoff 2 - norm orthogonality.

DEFINITION 3. If \((x, z), (y, z) \in X \times X\), we say \((x, z)\) is Birkhoff orthogonal to \((y, z)\), denoted \((x, z) \perp_{\beta} (y, z)\) provided \(\| x + ky, z \|_{L^p} \geq \| x, z \|_{L^p}\) for all real numbers \(k\).
THEOREM 5. Let $1 < p \leq 2$ and $0 < \alpha, \beta < 1$ be fixed real numbers. Then Birkhoff $2 - \text{norm orthogonality implies } (\alpha, \beta) - L^p 2 - \text{norm orthogonality in } X$ if and only if $X$ is an $2 - \text{inner product space and } p = 2$.

PROOF. Let $(x, z) \perp \beta (y, z)$. By assumption and homogeneity of Birkhoff $2 - \text{norm orthogonality we get,}$

$$
\| x + y, z \|^p = \| \alpha x + y, z \|^p + \| x + \beta y, z \|^p - \| \alpha x + \beta y, z \|^p
$$

$$
= (\| \alpha^2 x + y, z \|^p + \| \alpha x + \beta y, z \|^p - \| \alpha^2 x + \beta y, z \|^p)
$$

$$
+ (\| \alpha x + \beta y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha x + \beta y, z \|^p) - \| \alpha x + \beta y, z \|^p
$$

$$
= (\| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p) - \| \alpha x + \beta y, z \|^p + \| \alpha x + \beta y, z \|^p
$$

$$
= \| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p - \| \alpha x + \beta^2 y, z \|^p
$$

$$
+ (\| \alpha^2 x + \beta y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p)
$$

$$
= \| \alpha^2 x + y, z \|^p + \| x + \beta^2 y, z \|^p - \| \alpha^2 x + \beta y, z \|^p
$$

Thus by induction we see that $(x, z) \perp \beta (y, z).$ implies

$$
\| x + y, z \|^p = \| \alpha^n x + y, z \|^p + \| x + \beta^n y, z \|^p - \| \alpha^n x + \beta^n y, z \|^p
$$

for $n \geq 1$. In the limit this yields $(x, z) \perp_p (y, z).$ implies

$$
\| x + y, z \|^p = \| x, z \|^p + \| y, z \|^p \tag{A}
$$

If for $p = 2$ then (A) yields

$$
\| x + y, z \|^2 = \| x, z \|^2 + \| y, z \|^2 \tag{B}
$$

From the definition of $2 - \text{inner product space, we have}$

$$
\| x + y, z \|^2 = (x + y, x + y/z)
$$

and

$$
\| x, z \|^2 = (x, x/z)
$$

$$
\| y, z \|^2 = (y, y/z)
$$

492
From (B) \((x + y, x/z) + (x + y, y/z) = (x, x/z) + (y, y/z)\)
i.e. \((x, x/z) + y, x/z) + (y, y/z)(y, y/z)\)
\(= (x, x/z) + (y, y/z) + 2(x, y/z) = 0 \Rightarrow (x, y/z) = 0\)
Although in 1 - norm space the proof that identity (A) implies
\[\mu_p(X) = \sup_{(x,y) \in (y,z)} \frac{(\|x, x\|^p + (\|y, y\|/z)^2}{\|x + y, z\|^p} = 1\]
which in turn implies that X is an inner product space by the technique
of Amir[1]. But one has to explore whether the same proof will work for
1 < p ≤ 2 in the context of 2 - norm spaces.

THEOREM 6. Let 1 < p ≤ 2 and 0 < α, β < 1, be fixed real numbers. Then \((α, β) - L^p\) 2 - norm orthogonality implies Birkhoff 2 - norm orthogonality in X if and only if X is an inner product space and p = 2.

PROOF. We first prove that if \((α, β) - L^p\) 2 - norm orthogonality implies Birkhoff orthogonality then X is strict convex. If not then we can choose \(x \neq y\) as extreme points of the units ball of X such that \(\|x, z\| = \|y, z\|\)
\(= \|\frac{x + y}{z}, z\| = 1.\)
Then
\[\|\frac{x + y}{z} + y, z\|^p + \|\frac{αx + y}{z} + βy, z\|^p = \|\frac{x + y}{z} + y, z\|^p + \|\frac{x + y}{z} + βy, z\|^p\]
For otherwise \(2^p + (α + β)^p = (α + 1)^p + (β + 1)^p\) which requires \(α = 1\) or \(β = 1\), i.e. \((\frac{α + y}{z}, z)\) is not \((α, β) - L^p\) 2 - norm orthogonal to \((y, z)\). Without loss of generality we assume \(α ≥ β\). By Theorem 2 we can choose \(α ≠ 0\) such that \((\frac{α + y}{z}, z) = L^p \frac{α + y}{z} + (y, z))\(α + β\).
Hence \((\frac{α + y}{z}, z) = L^p \frac{α + y}{z} + (y, z))\) i.e. \(\|\frac{α + y}{z} + k(α ± y + y), z\| ≥ \|\frac{α + y}{z}, z\| = 1\) for all real numbers k. Putting
\(k = -1/2\) yields \(|α| ≤ 1,\) and then \(k = -1/α^2\) yields \(|α + 2| ≤ 1\). Thus \(α = -1\).
But then \((\frac{α + y}{z}, z) = L^p (\frac{α + y}{z} + (y, z))\) and then it gives
\[1 + \|\frac{α - β}{z} x + \frac{α + β}{z} y, z\|^p = \|\frac{α - 1}{z} x + \frac{α + 1}{z} y, z\|^p + \|\frac{1 - β}{z} x + \frac{1 + β}{z} y, z\|^p\]
So we have
\[α^p = α^p \|\frac{α - β}{2α} x + \frac{α + β}{2α} y, z\|^p = \|\frac{α - 1}{z} x + \frac{α + 1}{z} y, z\|^p\]
and it follows \( \| \frac{\alpha-1}{2\alpha} x + \frac{\alpha+1}{2\alpha} y, z \| = 1. \)

Writing \((y, z) = \frac{1}{1+\alpha}(x) + (1 - \frac{\alpha-1}{\alpha+1}) (\frac{\alpha-1}{2\alpha} x + \frac{\alpha+1}{2\alpha} y).\)

We see that \(y\) is a convex combination of two points of the unit sphere which is false since \(y\) was taken to be an extrem point of the unit ball \(X\).

Thus \(X\) must be strictly convex.

Now we prove that if \((\alpha, \beta) - \text{L}_p\) 2 - norm orthogonality implies \((\alpha, \beta) - \text{L}_p\) 2 - norm orthogonality. If not, then there exists \((x, z)(y, z) \in X \times X\) such that \((x, z) \perp_\beta (y, z)\) and \((x, x)\) is not \((\alpha, \beta) - \text{L}_p\) 2 - norm orthogonality to \((y, z)\). By Corollary 1 we can choose \(b \neq 0\) such that \((by + x, z) \perp_{L_p} (y, z)) (\alpha, \beta)\). But then \((by + x, z) \perp_\beta (y, z)\). Thus we have \((x, z) \perp_\beta (y, z)\) and \((by + x, z) \perp_\beta (y, z)\) which contradicts the left uniqueness of \(2 - \text{norm}\) Birkhoff orthogonality in strict convex space [5], hence \(2 - \text{norm}\) Birkhoff orthogonality implies \((\alpha, \beta) - \text{L}_p\) 2 - norm orthogonality, which is sufficient for \(X\) to be an inner product space and \(p = 2\) by Theorem 5.

The other part is also obvious.

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