COMMON FIXED POINTS FOR FOUR NON-SELF-MAPPINGS IN PARTIAL METRIC SPACES

Terentius Rugumisa, Santosh Kumar and Mohammad Imdad

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Abstract. In this paper, we formulate a common fixed point theorem for four non-self mappings in convex partial metric spaces. The result extends a fixed point theorem by Gajić and Rakočević [Pair of non-self mappings and common fixed points. Appl. Math. Comp. 187 (2007), 999-1006] proved for two non-self mappings in metric spaces with a Takahashi convex structure. We also provide an illustrative example on the use of the theorem.

Keywords: Common fixed point, convex partial metric space, non-self mapping.

MSC 2010: 47H10, 54H25

1. Introduction and Preliminaries

Gajić and Rakočević [5] proved a common fixed point theorem for non-self mappings on a Takahashi convex metric space for a pair of mappings. In their work, they generalized the theorems by Jungck [7], Das and Naik [4], Ćirić et al. [3], Ćirić [2] and Imdad and Kumar [6]. In this study, we extend the theorem by Gajić and Rakočević to apply for two pairs of non-self mappings in convex partial metric spaces.

We now introduce those results which will be of use in this paper.

Definition 1.1. [8] A partial metric on a non-empty set $X$ is a mapping $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(P0): $0 \leq p(x, x) \leq p(x, y)$,
(P1): $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(P2): $p(x, y) = p(y, x)$ and
(P3): $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A pair $(X, p)$ is said to be a partial metric space.
From Definition 1.1, we deduce that for all $x, y, z$ in a partial metric space $(X, p)$, we have:

(1.1) \hspace{1cm} (i) \quad p(x, y) = 0 \text{ implies } x = y,
(1.2) \hspace{1cm} (ii) \quad p(x, y) \leq p(x, z) + p(z, y).

Proof. If $p(x, y) = 0$, then $p(x, x) = 0$ because $0 \leq p(x, x) \leq p(x, y)$ from (P0). Similarly, $p(x, y) = 0$ implies $p(y, y) = 0$ because $0 \leq p(y, y) \leq p(x, y)$. Hence $p(x, y) = 0$ implies $p(x, x) = p(x, y) = p(y, y) = 0$. From (P1) this means that $x = y$.

From (P3), we infer that $p(x, y) \leq p(x, z) + p(z, y)$.

As an example, let $X = \mathbb{R}^+$ and let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x, y) = \max\{x, y\}$. Then $(X, p)$ is a partial metric space.

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base being the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space $(X, p)$ converges to $x \in X$ if and only if

$$p(x, x) = \lim_{n \to \infty} p(x, x_n).$$

Definition 1.2. [8] Let $(X, p)$ be a partial metric space and $\{x_n\}$ be a sequence in $X$. Then

(i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$,
(ii) $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m \to +\infty} p(x_n, x_m)$ exists and finite,
(iii) A partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that

$$p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m).$$

Lemma 1.3. [8] If $p$ is a partial metric on $X$, then the mapping $p^* : X \times X \to \mathbb{R}^+$ given by

$$p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric.

In this paper we will denote $p^*$ as the metric derived from the partial metric $p$. 
Lemma 1.4. [8] Let $(X, p)$ be a partial metric space and $\{x_n\}$ be a sequence in $X$. Then

(i) $\{x_n\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, p^*)$.

(ii) $(X, p)$ is complete if and only if $(X, p^*)$ is complete. Furthermore, $\lim_{n \to \infty} p(x_n, x)$ is zero if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, x_m)$.

We define $0$-complete partial metric spaces.

Definition 1.5. [8] Let $(X, p)$ be a partial metric space and $\{x_n\}$ be a sequence in $X$. Then

(i) the sequence $\{x_n\}$ is called $0$-Cauchy if $\lim_{n, m \to \infty} p(x_n, x_m) = 0$, 

(ii) $(X, p)$ is said to be $0$-complete if every $0$-Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = 0$.

We define a convex partial metric space.

Definition 1.6. [10] Let $(X, p)$ be a partial metric space and $I = [0, 1]$ be the closed unit interval. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on $X$ if for all $(x, y, t) \in X \times X \times I$,

$$p(u, W(x, y, t)) \leq tp(u, x) + (1 - t)p(u, y)$$

for every $u \in X$. A partial metric space $(X, p)$, together with the convex structure $W$, is called a convex partial metric space.

If $(X, p)$ is a convex partial metric space, then for every $x, y \in X$, we define

$$(1.3) \quad \text{seg}[x, y] := \{W(x, y, t) : t \in [0, 1]\}.$$

In this study, we will use the following properties for a convex partial metric space with convex structure $W$.

Lemma 1.7. Let $x, y \in X$ where $(X, p)$ is a convex partial metric space with convex structure $W$. Let $w \in \text{seg}[x, y]$. Then for all $u \in X$, we have

(i) $p(u, w) \leq \max\{p(u, x), p(u, y)\}$,

(ii) $p(x, w) \leq p(x, y)$.

Proof. Suppose $\Gamma = \max\{p(u, x), p(u, y)\}$. Applying Definition 1.6, we have

$$p(u, w) \leq tp(u, x) + (1 - t)p(u, y)$$

$$\leq t\Gamma + (1 - t)\Gamma$$

$$= \Gamma$$

$$= \max\{p(u, x), p(u, y)\}.$$
We have proved Lemma 1.7 (i). Now let us set \( x = u \) in Lemma 1.7 (i). We get

\[
p(x, u) \leq \max\{p(x, x), p(x, y)\} = p(x, y),
\]
from P0 of Definition 1.1.

\[\square\]

**Definition 1.8.** [1] Let \((X, p)\) be a partial metric space and \( B \subseteq X \). Then

(i) \( B \) is said to be bounded if there is a positive number \( M \) such that \( p(x, y) \leq M \) for all \( x, y \in B \),

(ii) if \( B \) is a bounded set, the diameter of \( B \) is defined as

\[
diam(B) = \sup_{u, v \in B} \{p(u, v)\}.
\]

Let \( f : C \to X \) be a mapping, where \( C \subseteq X \). We say that \( f \) is a self mapping if \( C = X \), otherwise \( f \) is called a non-self mapping. If there is an element \( x \in C \) such that \( fx = x \), we say that \( x \) is a fixed point of \( f \) in \( X \).

Suppose we have two mappings \( f, g : C \to X \), with \( C \subseteq X \). Let there be \( x \in C \) such that \( fx = gx = w \). We say that \( x \) is a coincidence point of \( f \) and \( g \) in \( X \). If \( x = w \), then we call \( x \) a common fixed point of \( f \) and \( g \) in \( X \).

Suppose we have two mappings \( f, g : C \to X \) with \( C \subseteq X \). We say \( f \) and \( g \) are coincidentally commuting if for all \( x \in C \), we have

\[fx = gx \Rightarrow gfx = gfx.\]

In this paper, we aim to extend the following theorem by Gajić and Rakočević [5] which proves the existence of a common fixed point for non-self mappings in context of metric spaces under specified conditions.

**Theorem 1.9.** [5] Let \((X, d)\) be a complete Takahashi convex metric space with convex structure \( W \) which is continuous in the third variable. Let \( C \) be a non-empty closed subset of \( X \) and \( \partial C \) be the boundary of \( C \). Let \( f, g : C \to X \) and suppose \( \partial C \neq \emptyset \). Let us assume that \( f \) and \( g \) satisfy the following conditions:

(i) For every \( x, y \in C \), \( d(gx, gy) \leq M_\omega(x, y) \) where \( M_\omega(x, y) = \max\{\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)], \omega_4[d(fx, gy)]\} \), \( \omega_i : [0, +\infty) \to [0, +\infty) \), \( i \in \{1, 2, 3, 4, 5\} \) is a non-decreasing semicontinuous function from the right, such that \( \omega_i(r) < r \) for \( r > 0 \), and \( \lim_{r \to \infty}[r - \omega_i(r)] = +\infty \),

(ii) \( \partial C \subseteq f(C) \),

(iii) \( g(C) \cap C \subset f(C) \),

(iv) \( fx \in \partial C \Rightarrow gx \in C \) and

(v) \( f(C) \) is closed in \( X \).
Then there exists a coincidence point \( v \) in \( C \). Moreover, if \( \{ f, g \} \) are coincidentally commuting, then \( v \) remains a unique common fixed point of \( f \) and \( g \).

We now proceed to the main results.

2. Main Results

In this section, we extend Theorem 1.9 to two pairs of non-self mappings. We prove the following assumption:

**Theorem 2.1.** Let \((X, p)\) be a complete convex partial metric space with convex structure \( W \) which is continuous in the third variable. Let \( C \) be a closed subset of \( X \) with a non-empty boundary \( \partial C \). Let 
\[ S, T, A, B : C \to X. \]
Let us assume that \( S, T, A \) and \( B \) satisfy the following conditions:

(i) For every \( x, y \in C \), \( p(Ax, By) \leq M_\omega(x, y) \) where
\[ M_\omega(x, y) = \max \{ \omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)], \omega_4[p(Ax, Ty)], \omega_5[p(Sx, By)] \}, \]
\( \omega_i : [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5, \) is a non-decreasing semicontinuous function from the right, such that \( \omega_i(r) < r/2 \) for \( r > 0 \), and
\[ \lim_{r \to +\infty} [r - 2\omega_i(r)] = +\infty. \]

(ii) \( \partial C \subseteq TC, \partial C \subseteq SC \),
(iii) \( Sx \in \partial C \Rightarrow Ax \in C \); \( Tx \in \partial C \Rightarrow Bx \in C \),
(iv) \( AC \cap C \subseteq TC, BC \cap C \subseteq SC \) and
(v) \( SC, TC \) are closed in \( C \).

Then there exists a coincidence point \( z \in C \) for \( A, B, S \) and \( T \). Moreover, if each of the pairs \( \{ S, A \} \) and \( \{ T, B \} \) is coincidentally commuting, then \( z \) remains a unique common fixed point of \( A, B, S \) and \( T \).

**Proof.** Commencing with an arbitrary point \( w \in \partial C \), we construct a sequence \( \{ x_n \} \) of points in \( C \) as follows:

From assumption (ii), there is a point \( x_0 \in C \) such that \( Sx_0 = w \). We find \( Ax_0 \).

Then we proceed inductively as follows.

If \( Ax_{2n} \in C \), then, by (iv), we choose \( x_{2n+1} \in C \) such that \( Tx_{2n+1} = Ax_{2n} \).

If however \( Ax_{2n} \notin C \), because \( W \) is continuous in the third variable, it means that, by (iii), there is \( \lambda_{2n, 2n} \in (0, 1) \) such that
\[ W(Sx_{2n}, Ax_{2n}, \lambda_{2n, 2n}) \in \partial C. \]

By (ii), this means we can choose \( x_{n+1} \in C \) such that
\[ Tx_{n+1} = W(Sx_{2n}, Ax_{2n}, \lambda_{2n, 2n}) \in \partial C. \]

By the construction, \( x_{n+1} \) belongs to \( C \) for all \( n \). Hence, \( \{ x_n \} \) is a Cauchy sequence in \( C \).

Since \( \partial C \subseteq TC, \partial C \subseteq SC \), \( \{ x_n \} \) converges in \( C \) to a point \( z \), which is a coincidence point of \( S, T, A \) and \( B \).
We then determine $Bx_{2n+1}$.

If $Bx_{2n+1} \in C$, then, by (iv), we choose $x_{2n+2} \in C$ such that $Sx_{2n+2} = Bx_{2n+1}$.

However if $Bx_{2n+1} \notin C$, because $W$ is continuous in the third variable, this means there is $\lambda_{2n+1,2n+1} \in (0,1)$ such that

$$W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}) \in \partial C.$$ 

By (ii), this means we can choose $x_{n+2} \in C$ such that

$$Sx_{n+2} = W(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}).$$

We then determine $Ax_{2n+2}$.

We show that, for $n \geq 1$, we have

$$(2.1) \quad Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_{2n}.$$ 

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \partial C$, which by (iii) means $Ax_{2n} \in C$. This, by (iv), implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction.

Using a similar argument we have

$$(2.2) \quad Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1}.$$ 

We now prove that the sequences $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geq 1$ let

$$D_n = \left( \bigcup_{i=0}^{n-1} \{Ax_{2i}\} \right) \cup \left( \bigcup_{i=0}^{n-1} \{Bx_{2i+1}\} \right) \cup \left( \bigcup_{i=0}^{n-1} \{Sx_{2i}\} \right) \cup \left( \bigcup_{i=0}^{n-1} \{Tx_{2i+1}\} \right),$$

$n \geq 1$.

Let $\alpha_n = \text{diam}(D_n)$. We show that

$$(2.3) \quad \alpha_n \leq \max \{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, 0 \leq j \leq n - 1.$$ 

Let us consider the case where $\alpha_n = 0, n \geq 1$.

This means $p(Sx_0, Ax_0) = p(Ax_0, Bx_1) = p(Bx_1, Tx_1) = 0$. Applying (1.2) this means

$$Sx_0 = Ax_0 = Bx_1 = Tx_1.$$ 

We shall show that $Sx_0$ is a common fixed point of $S$ and $A$. As the mappings $S$ and $A$ are coincidentally commuting at the coincidence point $x_0$, we have

$$(2.4) \quad Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0.$$
From (i), we have
\[ p(SSx_0, Sx_0) = p(ASx_0, Bx_1) \leq M_\omega (Sx_0, x_1) \]
\[ = \max \{ \omega_1 [p(SSx_0, Tx_1), \omega_2 [p(ASx_0, SSx_0) | \omega_3 [p(Bx_1, Tx_1)], \omega_4 [p(ASx_0, Tx_1)], \omega_5 [p(SSx_0, Bx_1)] \} \]
\[ = \max \{ \omega_1 [p(SSx_0, Sx_0)], \omega_2 [p(SSx_0, SSx_0)], \omega_3 [p(Sx_0, Sx_0)], \omega_4 [p(SSx_0, Sx_0)], \omega_5 [p(SSx_0, Sx_0)] \} \]
\[ \leq \omega_t [p(SSx_0, Sx_0)], \text{for some } t \in \{1, 2, 3, 4, 5\}, \]
\[ < \frac{1}{2} p(SSx_0, Sx_0), \text{for } p(SSx_0, Sx_0) > 0 \]
\[ \Rightarrow p(SSx_0, Sx_0) = 0. \]

By (1.2), this implies
\[ (2.5) \quad SSx_0 = Sx_0. \]

Hence $Sx_0$ is a fixed point of $S$. From (2.4) we have $SSx_0 = ASx_0$. Thus (2.5) implies $ASx_0 = Sx_0$, making $Sx_0$ a fixed point of $A$ too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of $T$ and $B$. Hence, $z = Sx_0$ is a common fixed point of all four mappings $S, T, A$ and $B$.

To show the uniqueness of the fixed point, let $z'$ be also a fixed point of $S, T, A$ and $B$. Then we have
\[ p(z, z') = p(Az, Bz') \]
\[ \leq \max \{ \omega_1 [p(z, Tz'), \omega_2 [p(Az, Sz)], \omega_3 [p(Bz', Tz')] \}, \omega_4 [p(Az, Tz')], \omega_5 [p(Sz, Bz')] \} \]
\[ = \max \{ \omega_1 [p(z, z')], \omega_2 [p(z, z)], \omega_3 [p(z', z')], \omega_4 [p(z, z')], \omega_5 [p(z, z')] \} \]
\[ \leq \omega_i [p(z, z')] \text{ for some } i \in \{1, 2, 3, 4, 5\} \]
\[ < \frac{1}{2} p(z, z'), \text{for } p(z, z') > 0, \]
\[ \Rightarrow p(z, z') = 0 \]
\[ \Rightarrow z = z, \text{ by (1.1)}. \]

Hence when $\alpha_n = 0$, $z = Sx_0$ is the unique common fixed point of $S, T, A$ and $B$.

We now consider the cases when $\alpha_n > 0$.

**Case 1:** Consider the case where $\alpha_n = p(Sx_{2i}, Ax_{2j})$ for some $0 \leq i, j \leq n - 1$. 

(1. i) If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ we have

\[ \alpha_n = p(Sx_{2i}, Ax_{2j}) = p(Ax_{2j}, Bx_{2i-1}) \]
\[ \leq M_w(x_{2j}, x_{2i-1}) \]
\[ \leq \omega_s(\alpha_n) \text{ for some } s \in \{1, 2, 3, 4, 5\} \]
\[ < \frac{1}{2}\alpha_n, \]

which is a contradiction. Hence $i = 0$.

(1.ii) If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it implies $Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and hence, by Lemma 1.7 (i), we have

\[ \alpha_n = p(Sx_{2i}, Ax_{2j}) \leq \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i-2}, Ax_{2j})\}. \]

(1.ii.1) If $p(Ax_{2j}, Bx_{2i-1}) \geq p(Ax_{2i-2}, Ax_{2j})$, we have

\[ \alpha_n = p(Sx_{2i}, Ax_{2j}) \leq p(Ax_{2j}, Bx_{2i-1}), \]

which leads to the contradiction in (1. i), meaning that $i = 0$.

(1.ii.2) Otherwise, if $p(Ax_{2j}, Bx_{2i-1}) < p(Ax_{2i-2}, Ax_{2j})$, then for some $k$ such that $2i - 2 < 2k + 1 < 2j$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have

\[ \alpha_n = p(Sx_{2i}, Ax_{2j}) \leq p(Ax_{2i-2}, Ax_{2j}) \]
\[ \leq p(Ax_{2i-2}, Bx_{2k+1}) + p(Ax_{2j}, Bx_{2k+1}), \text{ by (1.2)} \]
\[ \leq M_w(x_{2i-2}, x_{2i-1}) + M_w(x_{2j}, x_{2i-1}) \]
\[ \leq \omega_s(\alpha_n) + \omega_t(\alpha_n) \]
\[ < \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \]
\[ \Rightarrow \alpha_n < \alpha_n, \]

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = p(Ax_{2i}, Bx_{2j+1})$ leads to contradiction by (1. i).

Case 3: The case where $\alpha_n = p(Ax_{2i}, Ax_{2j})$ leads to contradiction by (1.ii.2).
Case 4: If $\alpha_n = p(Bx_{2i+1}, Bx_{2j+1})$ then for $k$ such that $2i + 1 < 2k < 2j + 1$ and for some $s, t \in \{1, 2, \ldots, 5\}$, we have
\[
\alpha_n = p(Bx_{2i+1}, Bx_{2j+1}) \\
\leq p(Ax_{2k}, Bx_{2i+1}) + p(Ax_{2k}, Bx_{2j+1}) \\
\leq M_\omega(x_{2k}, x_{2i+1}) + M_\omega(x_{2k}, x_{2j+1}) \\
\leq \omega_s(\alpha_n) + \omega_t(\alpha_n) \\
< \frac{1}{2} \alpha_n + \frac{1}{2} \alpha_n \\
\Rightarrow \alpha_n < \alpha_n,
\]
which is a contradiction.

Case 5: If $\alpha_n = p(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \leq i, j \leq n - 1$, then:

(5.i) If $Tx_{2i+1} = Ax_{2i}$, then we have
\[
\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) = p(Ax_{2i}, Bx_{2j+1}),
\]
which is a contradiction by (1.i).

(5.ii) Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2j}]$ and hence by Lemma 1.7 (i) we have
\[
\alpha_n = p(Tx_{2i+1}, Bx_{2j+1}) \leq \max\{p(Bx_{2i-1}, Bx_{2j+1}), p(Ax_{2i}, Bx_{2j+1})\}.
\]
This means we have either $p(Tx_{2i+1}, Bx_{2j+1}) \leq p(Bx_{2i-1}, Bx_{2j+1})$, which is a contradiction by Case 4 or $p(Tx_{2i+1}, Bx_{2j+1}) \leq p(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by (1.i).

Case 6: If $\alpha_n = p(Tx_{2i+1}, Ax_{2j})$ for some $0 \leq i, j \leq n - 1$, then:

(6.i) If $Tx_{2i+1} = Ax_{2i}$, then we have $\alpha_n = p(Tx_{2i+1}, Ax_{2j}) = p(Ax_{2i}, Ax_{2j})$ which is not possible by (1.ii.2)

(6.ii) Otherwise, if $Tx_{2i+1} \neq Ax_{2i}$, then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2j}]$ and hence $\alpha_n = p(Tx_{2i+1}, Ax_{2j}) \leq \max\{p(Ax_{2j}, Bx_{2i-1}), p(Ax_{2i}, Ax_{2j})\}$. This implies we have either $p(Tx_{2i+1}, Ax_{2j}) \leq p(Ax_{2j}, Bx_{2i-1})$, which is a contradiction by (1.i) or else we have $p(Tx_{2i+1}, Ax_{2j}) \leq p(Ax_{2i}, Ax_{2j})$, which is a contradiction by (1.ii.2).

Case 7: Suppose $\alpha_n = p(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \leq i, j \leq n - 1$.

(7.i) If $Tx_{2j+1} = Ax_{2j}$, we have $\alpha_n = p(Tx_{2i+1}, Tx_{2j+1}) = p(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

(7.ii) Otherwise if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \text{seg}[Bx_{2j-1}, Ax_{2j}]$ and hence $\alpha_n = p(Tx_{2i+1}, Tx_{j+1}) \leq \max\{p(Tx_{2i+1}, Bx_{2j-1}), p(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $p(Tx_{2i+1}, Tx_{2j+1}) \leq p(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5 or else we have $p(Tx_{2i+1}, Tx_{2j+1}) \leq p(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.
Case 8: Let $\alpha_n = p(Sx_{2i}, Bx_{2j+1})$, for some $0 \leq i, j \leq n - 1$.

(8.i) If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$, then
$\alpha_n = p(Sx_{2i}, Bx_{2j+1}) = p(Bx_{2i-1}, Bx_{2j+1})$ which is not possible as per Case 4. Hence $i = 0$.

(8.ii) If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it means that
$Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$. This implies that
$\alpha_n = p(Sx_{2i}, Bx_{2j+1}) \leq \max\{p(Ax_{2i-2}, Bx_{2j+1}), p(Bx_{2i-1}, Bx_{2j+1})\}$.

This leads to a contradiction by (1.i) when
$p(Sx_{2i}, Bx_{2j+1}) \leq p(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens
that $p(Sx_{2i}, Bx_{2j+1}) \leq p(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: Let us consider when $\alpha_n = p(Sx_{2i}, Sx_{2j})$, for some $0 \leq i < j \leq n - 1$.

(9.i) If $i \geq 1$ and $Sx_{2j} = Bx_{2j-1}$ then we have
$\alpha_n = p(Sx_{2i}, Sx_{2j}) = p(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to
Case 8. Hence $i = 0$.

(9.ii) If $i \geq 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it implies that $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$ and
$p(Sx_{2i}, Sx_{2j}) \leq \max\{p(Sx_{2i}, Ax_{2j-2}), p(Sx_{2i}, Bx_{2j-1})\}$.

If it happens $p(Sx_{2i}, Sx_{2j}) \leq p(Sx_{2i}, Ax_{2j-2})$, we get a contradiction by Case 1. However if it happens
that $p(Sx_{2i}, Sx_{2j}) \leq p(Sx_{2i}, Bx_{2j-1})$, then we get a contra-
diction by Case 8. Hence $i = 0$.

Case 10: Suppose $\alpha_n = p(Sx_{2i}, Tx_{2j+1})$, for some $0 \leq i, j \leq n - 1$.

(10.i) If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$, then we have
$\alpha_n = p(Sx_{2i}, Tx_{2j+1}) = p(Tx_{2j+1}, Bx_{2i-1})$ which is not possible as per Case 8. Hence $i = 0$.

(10.ii) If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it implies that
$Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and
$p(Sx_{2i}, Tx_{2j+1}) \leq \max\{p(Tx_{2j+1}, Ax_{2i-2}), p(Tx_{2j+1}, Bx_{2i-1})\}$. This leads to con-
tradictions by Case 6 and Case 5. Hence $i = 0$.

We have considered ten possible cases for $\alpha_n$ and conclude from Cases 1, 8, 9 and
10 that for some $0 \leq j \leq n - 1$ we have

(2.6) $\alpha_n \in \{p(Sx_0, Sx_{2j}), p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1}), p(Sx_0, Tx_{2j+1})\}$.

Note that, from the construction of sequence, $Sx_{2j} \in C$ implies
$Sx_{2j} = Bx_{2j-1}$. This leads to

(2.7) $p(Sx_0, Sx_{2j}) = p(Sx_0, Bx_{2j-1})$.

For $Sx_{2j} \notin C$, we have $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$. From Lemma 1.7 (i), this implies

(2.8) $p(Sx_0, Sx_{2j}) \leq \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}$.
From (2.7) and (2.8) we conclude that

\[(2.9) \quad p(Sx_0, Sx_{2j}) \leq \max\{p(Sx_0, Ax_{2j-2}), p(Sx_0, Bx_{2j-1})\}.\]

Using a similar argument we also have

\[(2.10) \quad p(Sx_0, Tx_{2j+1}) \leq \max\{p(Sx_0, Bx_{2j+1}), p(Sx_0, Ax_{2j})\}.\]

Applying (2.9) and (2.10) to (2.6) we get

\[\alpha_n \leq \max\{p(Sx_0, Ax_{2j}), p(Sx_0, Bx_{2j+1})\}, \quad 0 \leq j \leq n - 1.\]

We have proved (2.3).

Consider the case where \(\max\{p(Sx_0, Ax_{2j})\} \leq \max\{p(Sx_0, Bx_{2j+1})\}, \quad 0 \leq j \leq n - 1.\) Then, for some \(u \in \{1, 2, \ldots, 5\}, (2.3)\) implies

\[\alpha_n \leq p(Sx_0, Ax_{2j+1})\]
\[\leq p(Sx_0, Ax_0) + p(Ax_0, Bx_{2j+1}), \text{ by (1.2)}\]
\[\leq p(Sx_0, Ax_0) + \omega_u[\alpha_n]\]
\[\leq p(Sx_0, Ax_0) + 2\omega_u[\alpha_n]\]
\[\Rightarrow \alpha_n - 2\omega_u[\alpha_n] \leq p(Sx_0, Ax_0).\]

Consider now, when \(\max\{p(Sx_0, Ax_{2j})\} > \max\{p(Sx_0, Bx_{2j+1})\}, \quad 0 \leq j \leq n - 1.\) Then for some \(v \in \{1, 2, \ldots, 5\}, (2.3)\) implies

\[\alpha_n \leq p(Sx_0, Ax_{2j})\]
\[\leq p(Sx_0, Ax_0) + p(Ax_0, Ax_{2j}), \text{ by (1.2)}\]
\[\leq p(Sx_0, Ax_0) + 2\omega_v[\alpha_n], \text{ by (1.ii.2)}\]
\[\Rightarrow \alpha_n - 2\omega_v[\alpha_n] \leq p(Sx_0, Ax_0).\]

Thus in both cases, we have for some \(s \in \{1, 2, \ldots, 5\}\)

\[(2.11) \quad \alpha_n - 2\omega_s[\alpha_n] \leq p(Sx_0, Ax_0).\]

By assumption (i) there is an \(r_0 \in [0, +\infty)\) such that for each \(s \in \{1, 2, \ldots, 5\},\)
\[r - 2\omega_s[r] > p(Sx_0, Ax_0)\text{ for } r > r_0.\]

There is a subsequence \(\{a_n\}\) of \(\{\alpha_n\}\) and \(s \in \{1, 2, \ldots, 5\}\) such that for each \(n,\)
we have
\[a_n - 2\omega_s[a_n] \leq p(Sx_0, Ax_0).\]
Thus by (2.11), \( a_n \leq r_0 \). Thus we have

\[
a = \lim_{n \to +\infty} a_n = \text{diam}(D) \leq r_0.
\]

We have hence proved that \( \{Sx_{2n}\}, \{Tx_{2n+1}\}, \{Ax_{2n}\} \) and \( \{Bx_{2n+1}\} \) are bounded sequences.

To prove that \( \{Sx_{2n}\}, \{Tx_{2n+1}\}, \{Ax_{2n}\} \) and \( \{Bx_{2n+1}\} \) converge in \( C \), we reflect on the set

\[
(2.12) \quad E_n = \left( \bigcup_{i=n}^{+\infty} \{Ax_i\} \right) \cup \left( \bigcup_{i=n}^{+\infty} \{Bx_{2i+1}\} \right) \cup \left( \bigcup_{i=n}^{+\infty} \{Sx_i\} \right) \cup \left( \bigcup_{i=n}^{+\infty} \{Tx_{2i+1}\} \right),
\]

\( n = 2, 3, \ldots \)

By (2.3) we have for \( n = 2, 3, \ldots \)

\[
(2.13) \quad e_n := \text{diam}(E_n) \leq \sup_{j \geq n} \{p(Sx_{2n}, Ax_j), p(Sx_{2n}, Bx_{2j+1})\}.
\]

If \( Sx_{2n} = Bx_{2n-1} \) we have as in Case 1 and Case 8, for each \( j \geq n \), and for some \( u \in \{1, 2, \ldots, 5\} \)

\[
e_n \leq \sup_{j \geq n} \{p(Ax_j, Bx_{2n-1}), p(Bx_{2j+1}, Bx_{2n-1})\}, n = 2, 3, \ldots
\]

\[
(2.14) \quad \leq 2\omega_u[e_{n-1}].
\]

If however \( Sx_{2n} \neq Bx_{2n-1} \), it implies that \( Sx_{2n} \in \text{seg}[Ax_{2n-2}, Bx_{2n-1}] \). Hence, as in Case 1 and Case 8, for each \( j \geq n \) and for some \( v \in \{1, 2, \ldots, 5\} \), we have

\[
e_n \leq \sup_{j \geq n} \{p(Ax_{2n-2}, Ax_j), p(Bx_{2n-1}, Ax_j), p(Ax_{2n-2}, Bx_{2j+1}), p(Bx_{2n-1}, Bx_{2j+1})\}, n = 2, 3, \ldots
\]

\[
(2.15) \quad \leq 2\omega_v[e_{n-2}].
\]

By (2.14) and (2.15), there is a subsequence \( \{\varepsilon_n\} \) of \( \{e_n\} \) and \( s \in \{1, 2, \ldots, 5\} \) such that for each \( n \)

\[
\varepsilon_n \leq 2\omega_s[e_{n-2}], n = 2, 3, \ldots
\]

\[
(2.16) \quad < \varepsilon_{n-2}.
\]

We note that \( e_n \geq e_{n+1} \) for every \( n \). Let \( \lim_{n \to +\infty} e_n = \lim_{n \to +\infty} \varepsilon_n = e \). We claim that \( e = 0 \).
If $e > 0$, then by (2.16) and assumption (i), we have

$$\lim_{n \to +\infty} \varepsilon_n < \lim_{n \to +\infty} \varepsilon_{n-2}$$

$$\Rightarrow e < e,$$

which is a contradiction. Hence $e = 0$.

We recall from (2.13) that $e_n = \text{Diam}(E_n)$. Taking $n, m \to +\infty$ in (2.12), we get

$$\lim_{n, m \to +\infty} p(Ax_{2n}, Ax_{2m}) = \lim_{n, m \to +\infty} p(Bx_{2n+1}, Bx_{2m+1}) = 0.$$

This means both $\{A_{2n}\}$ and $\{B_{2n+1}\}$ are Cauchy sequences.

Because $X$ is a complete partial metric space, this means there is $z \in X$ such that

$$\lim_{n \to +\infty} Ax_{2n} = \lim_{n \to +\infty} Bx_{2n+1} = z \text{ and } p(z, z) = 0.$$

Consider the subsequence $Sx_{2n_k}$ of $Sx_{2n}$ such that $Sx_{2n_k} = Bx_{2n_k-1}$. Taking $n_k \to +\infty$ we have

$$\lim_{n \to +\infty} Sx_{2n} = \lim_{n \to +\infty} Sx_{2n_k} = \lim_{n \to +\infty} Bx_{2n_k-1} = z, \text{ with } p(z, z) = 0.$$

Using a similar argument we have

$$\lim_{n \to +\infty} Tx_{2n+1} = z, \text{ with } p(z, z) = 0.$$

But $SC, TC$ are both 0-complete. This implies $z \in SC$ and $z \in TC$.

As $z \in SC$, there is a point $u \in C$ such that $Su = z$. We show that $u$ is a coincidence point of $A, B$ and $S$.

$$p(Au, Bx_{2n+1}) \leq \max\{\omega_1[p(Su, Tx_{2n+1})], \omega_2[p(Au, Su)],$$

$$\omega_3[p(Bx_{2n+1}, Tx_{2n+1})], \omega_4[p(Au, Tx_{2n+1})],$$

$$\omega_5[p(Su, Bx_{2n+1})]\}$$

$$= \max\{\omega_1[p(z, Tx_{2n+1})], \omega_2[p(Au, z)], \omega_3[p(Bx_{2n+1}, Tx_{2n+1})],$$

$$\omega_4[p(Au, Tx_{2n+1})], \omega_5[p(z, Bx_{2n+1})]\}.$$

Taking $n \to +\infty$ and applying (2.19) and (2.20), we get

$$p(Au, z) \leq \max\{\omega_1[p(z, z)], \omega_2[p(Au, z)], \omega_3[p(z, z)], \omega_4[p(Au, z)], \omega_5[p(z, z)]\}$$

$$\leq \omega_i[p(Au, z)] \text{ for some } i \in \{1, 2, \ldots, 5\}$$

$$< p(Au, z) \text{ for } p(Au, z) > 0$$

$$\Rightarrow p(Au, z) = 0$$

$$\Rightarrow Au = z, \text{ from (1.1).}$$
Using a similar procedure, when we expand \( p(Ax_{2n}, Bu) \), we get \( Bu = z \), making \( u \) a coincidence point of \( A, B \) and \( S \). By the coincidental commutativity of \( S \) and \( A \), we have \( SAu = ASu \Rightarrow Sz = Az \).

In the same vein, \( z \in TC \) means there is \( v \in C \), such that \( Tv = z \). We show that \( Bv = z \).

\[
p(z, Bv) = p(Au, Bv) \\
\leq \max\{\omega_1[p(Su, Tv)], \omega_2[p(Su, Au)], \omega_3[p(Tv, Bv)], \\
\omega_4[p(Au, Tv)], \omega_5[p(Su, Bv)]\} \\
= \max\{\omega_1[p(z, z)], \omega_2[p(z, z)], \omega_3[p(z, Bv)], \\
\omega_4[p(z, z)], \omega_5[p(z, Bv)]\} \\
\leq \omega_j[p(z, Bv)] \text{ for some } j \in \{1, 2, \ldots, 5\} \\
< p(z, Bv) \text{ for } p(z, Bv) > 0 \\
\Rightarrow p(z, Bv) = 0 \\
\Rightarrow Bv = z, \text{ from (1.1).}
\]

Thus \( v \) is a coincidence point of \( B \) and \( T \). By the coincidental commutativity property, we have

\begin{equation}
BTv = TBv \Rightarrow Bz = Tz.
\end{equation}

Now consider the following:

\[
p(Az, Bz) \leq \max\{\omega_1[p(Sz, Tz)], \omega_2[p(Sz, Az)], \omega_3[p(Tz, Bz)], \\
\omega_4[p(Az, Tz)], \omega_5[p(Sz, Bz)]\} \\
= \max\{\omega_1[p(Az, Bz)], \omega_2[p(Az, Az)], \omega_3[p(Bz, Bz)], \\
\omega_4[p(Az, Bz)], \omega_5[p(Az, Bz)]\} \\
\leq \omega_i[p(Az, Bz)] \text{ for } i \in \{1, 2, \ldots, 5\} \\
< p(Az, Bz) \text{ for } p(Az, Bz) > 0 \\
\Rightarrow p(Az, Bz) = 0 \\
\Rightarrow Az = Bz.
\]

Hence we have

\begin{equation}
(2.22) \quad Az = Bz = Sz = Tz.
\end{equation}
Now we consider the following:

\[ p(z, Bz) = p(Au, Bz), \text{ from (2.21)} \]
\[ \leq \max\{\omega_1[p(Su, Tz)], \omega_2[p(Au, Su)], \omega_3[p(Bz, Tz)], \omega_4[p(Au, Tz)], \omega_5[p(Su, Bz)]\} \]
\[ = \max\{\omega_1[p(z, Bz)], \omega_2[p(z, z)], \omega_3[p(Bz, Bz)], \omega_4[p(z, Bz)], \omega_5[p(z, Bz)]\}. \]

This implies

\[ p(z, Bz) \leq \omega_j[p(z, Bz)] \text{ for some } j \in \{1, 2, \ldots, 5\} \]
\[ < p(z, Bz) \text{ for } p(z, Bz) > 0 \]
\[ \Rightarrow p(z, Bz) = 0 \]
\[ \Rightarrow Bz = z \text{ by (1.1)}. \]

From (2.22) we conclude that
\[ Az = Bz = Sz = Tz = z, \]
meaning that \( z \) is a common fixed point of \( A, B, S \) and \( T \).

We now show that \( z \) is unique. Suppose \( z' \) is also a common fixed point of \( A, B, S \) and \( T \). We get

\[ p(z, z') = p(Az, Bz') \]
\[ \leq \max\{\omega_1[p(Sz, Tz')], \omega_2[p(Az, Sz)], \omega_3[p(Bz', Tz')], \omega_4[p(Az, Tz')], \omega_5[p(Sz, Bz')]\} \]
\[ = \max\{\omega_1[p(z, z')], \omega_2[p(z, z)], \omega_3[p(z', z')], \omega_4[p(z, z')], \omega_5[p(z, z')]\}. \]

This implies

\[ p(z, z') \leq \omega_k[p(z, z')] \text{ for } k \in \{1, 2, \ldots, 5\} \]
\[ < p(z, z') \text{ for } p(z, z') > 0 \]
\[ \Rightarrow p(z, z') = 0 \]
\[ \Rightarrow z = z'. \]

This proves that the common fixed point of \( A, B, S \) and \( T \) is unique. \( \square \)

**Remark 2.2.** Theorem 2.1 leads to corollaries if we set the following:
(i) $A = B$;
(ii) $A = B, S = T$, we get Theorem 1.9;
(iii) $A = B, S = T = I$, we get an extension of a theorem proved by Ćirić [2] into partial metric spaces;
(iv) $A = I$;
(v) $S = T$;
(vi) $x = y$.

The proof given works even when we define $C$ as closed in $(X,p^s)$. This lead to the following theorem.

**Theorem 2.3.** Let $(X, p)$ be a complete convex partial metric space with convex structure $W$ which is continuous in the third variable. Let $C$ be a closed subset of $X$, the closure being taken with respect to $(X,p^s)$. Let $\partial C$, the boundary of $C$ in $(X,p^s)$, be non-empty. Let $A, B, S, T : C \to X$. Let us assume that $A, B, S$ and $T$ satisfy the following conditions:

(i) For every $x, y \in C, p(Ax, By) \leq M_\omega(x, y)$ where $M_\omega(x, y) = \max\{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(By, Ty)], \omega_4[p(Ax, Ty)], \omega_5[p(Sx, By)]\}, \omega_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2, 3, 4, 5$, is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r/2$ for $r > 0$, and $\lim_{r \to +\infty}[r - 2\omega_i(r)] = +\infty$.

(ii) $\partial C \subseteq SC, \partial C \subseteq TC$,
(iii) $Sx \in \partial C \Rightarrow Ax \in C, Tx \in \partial C \Rightarrow Bx \in C$,
(iv) $AC \cap C \subset TC, BC \cap C \subset SC$ and
(v) $SC, TC$ are closed in $C$.

Then there exists a coincidence point $z \in C$ for $A, B, S$ and $T$. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then $z$ remains a unique common fixed point of $A, B, S$ and $T$.

Here we give an example on the use Theorem 2.3, as it is better suited for the partial metric that we will use.

**Example 2.4.** Consider the partial metric space $(\mathbb{R}_+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$. Let $C = [0, 2]$. We note that $C$ is closed in the derived metric $p^s(x, y) = |x - y|$ and $\partial C = \{0, 2\}$.

We define the mappings $A, B, S, T : C \to \mathbb{R}_+$ as follows:

\[
Ax = \begin{cases} 
3^x - 1, & x \in [0, 1] \\
1, & x \in (1, 2]
\end{cases}, \quad Bx = \begin{cases} 
4^x - 1, & x \in [0, 1] \\
2, & x \in (1, 2]
\end{cases},
\]

\[
Sx = \begin{cases} 
27^x - 1, & x \in [0, 1] \\
6, & x \in (1, 2]
\end{cases}, \quad Tx = \begin{cases} 
64^x - 1, & x \in [0, 1] \\
8, & x \in (1, 2]
\end{cases}.
\]
We have $AC = [0, 2]$ and $BC = [0, 3]$. We also have $TC = [0, 63]$ and $SC = [0, 26]$, both of which are closed in $(X, p_s)$.

$Sx \in \partial C$ implies $z \in \{0, 1/3\} \subset C$. Similarly $Tx \in \partial C$ implies $x \in \{0, \ln 3/\ln 64\} \subset C$. We also have $\partial K = \{1, 3\} \subseteq SC, TC$.

We note that $\{S, A\}$ and $\{T, B\}$ are both coincidentally commuting at $x = 0$: that is, $SA(0) = AS(0)$ and $TB(0) = BT(0)$. We also note that all four mappings are discontinuous at 1.

Let us define the functions $g, h : C \to \mathbb{R}_+$ as

$$g(x) = \frac{27^x - 1}{4^x - 1}, \quad h(x) = \frac{64^x - 1}{4^x - 1}.$$  

Both $g$ and $h$ are increasing functions. Using l’Hôpital rule, we can show that, as $x \to 0$, we have $h(x) \to 3$ and

$$g(x) \to \frac{\log 27}{\log 4} = \frac{1}{0.42062}. \quad (2.23)$$

Hence, for $x \in [0, 1]$, we have

$$4^x - 1 \leq \frac{1}{3} (64^x - 1) \leq 0.43(64^x - 1). \quad (2.24)$$

We also have, from (2.23)

$$4^x - 1 \leq 0.42062(27^x - 1) \quad (2.25) \Rightarrow 4^x - 1 \leq 0.43(27^x - 1).$$

When $x, y \in [0, 1]$, with $x \leq y$, we have

$$p(Ax, By) = p(3^x - 1, 4^y - 1) = \max\{3^x - 1, 4^y - 1\} = 4^y - 1, \text{ because } x \leq y \leq 0.43(64^y - 1), \text{ by } (2.24) = 0.43Ty \leq 0.43 \max\{Sx, Ty\} = 0.43p(Sx, Ty).$$
When $x, y \in [0, 1]$, with $x > y$, we have

$$p(Ax, By) = p(3^x - 1, 4^y - 1)$$

$$= \max\{4^x - 1, 4^y - 1\}, \text{ because } 3^x - 1 \leq 4^x - 1$$

$$= 4^x - 1, \text{ because } x > y$$

$$\leq 0.43(27^x - 1), \text{ by (2.25)}$$

$$= 0.43Sx$$

$$\leq 0.43 \max\{Sx, Ty\}$$

$$= 0.43p(Sx, Ty).$$

For $x, y \in (1, 2]$ we have

$$p(Ax, By) = p(1, 2)$$

$$= \max\{1, 2\}$$

$$= 2$$

$$< 0.43 \times 6$$

$$\leq 0.43 \max\{6, Ty\}$$

$$= 0.43 \max\{Sx, Ty\}, \text{ because } Sx = 6$$

$$= 0.43p(Sx, Ty).$$

Consider when $x \in [0, 1], y \in (1, 2]$ we get

$$p(Ax, By) = p(3^x - 1, 2)$$

$$= \max\{3^x - 1, 2\}$$

$$= 2, \text{ because } 3^x - 1 \leq 2 \text{ for } x \in [0, 1]$$

$$< 0.43 \times 8$$

$$= 0.43Ty \text{ because } Ty = 8$$

$$\leq 0.43 \max\{Sx, Ty\}$$

$$= 0.43p(Sx, Ty).$$

Consider when $x \in (1, 2], y \in (0, 1)$.

$$p(Ax, By) = p(1, 4^y - 1)$$

$$= \max\{1, 4^y - 1\}$$

$$= \begin{cases} 
1 & \text{if } y \in (0.5, 1] \\
6 & \text{otherwise}
\end{cases}$$
For $y \in [0, 0.5]$, we have

\[
p(Ax, By) = p(1, 4^y - 1) \\
= 1 \\
< 0.43 \times 6 \\
= 0.43Sx \text{ because } Sx = 6 \\
\leq 0.43 \max\{Sx, Ty\} \\
= 0.43p(Sx, Ty).
\]

For $y \in (0.5, 1]$ we have

\[
p(Ax, By) = p(1, 4^y - 1) \\
= 4^y - 1 \\
\leq 0.43(64^y - 1), \text{ by (2.24)} \\
= 0.43Ty \\
\leq 0.43 \max\{Sx, Ty\} \\
= 0.43p(Sx, Ty).
\]

Thus in all cases, for every $x, y \in C$ we have

\[
p(Ax, By) \leq 0.43p(Sx, Ty) \leq M_\omega(x, y),
\]

where

\[
M_\omega(x, y) = \max \{\omega_1[p(Sx, Ty)], \omega_2[p(Ax, Sx)], \omega_3[p(Ay, Ty)], \omega_4[p(Ax, Ty)], \omega_5[p(Sx, Ay)]\},
\]

with $\omega_1[r] = 0.43r$. We note that $\omega_1[r] = 0.43r < \frac{1}{2}r$.

As $r - 2\omega_1[r] = r - 2 \times 0.43r = 0.14r$, we also have

\[
\lim_{r \to +\infty} r - 2\omega_1[r] = \lim_{r \to +\infty} 0.14r = +\infty.
\]

Thus all the conditions of Theorem 2.3 are satisfied and 0 is the unique common fixed point of $A, B, S$ and $T$.

References


Authors' addresses:
Terentius Rugumisa, Faculty of Science, Technology and Environmental Studies, The Open University of Tanzania, Tanzania. e-mail: rterentius@gmail.com
Santosh Kumar, Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Tanzania. e-mail: draengar2002@gmail.com
Mohammad Imdad, Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India. e-mail: mhimdad@gmail.com